MA3209 Metric and Topological Spaces

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Reference books:

(1). Munkres, J.R. (2000). *Topology*. Upper Saddle River, Nj: Prentice Hall, Inc.

(2). Adhikari, A. and Adhikari, M. R. (2023). *Basic Topology 1: Metric Spaces and General Topology*. Springer.

Contents

1. Topological Spaces and Continuous Functions

1.1. *Topological Spaces*

Definition 1.1 (topology). Let X be a set. A topology on X is a collection $\mathcal{T}$ of subsets of X such that

(i) $\emptyset, X \in \mathcal{T}$

(ii) The arbitrary union of members of $\mathcal{T}$ belongs to $\mathcal{T}$, i.e.

$$\{U_\alpha\}_{\alpha\in I} \in \mathcal{T} \quad \text{implies} \quad \bigcup_{\alpha\in I} U_\alpha \in \mathcal{T}.$$

(iii) The intersection of any finite members of $\mathcal{T}$ belongs to $\mathcal{T}$, i.e.

$$\{U_1, \ldots, U_n\} \in \mathcal{T} \quad \text{implies} \quad \bigcap_{i=1}^{n} U_i \in \mathcal{T}.$$

If the above-mentioned three conditions are satisfied, we say that $(X, \mathcal{T})$ is a topological space. A subset $U \subseteq X$ is open if $U \in \mathcal{T}$.

Example 1.1 (trivial topology). Let X be any set and $\mathcal{T} = \{\emptyset, X\}$. Then, $\mathcal{T}$ is called the trivial topology of X.

Example 1.2 (discrete topology). Let X be any set and $\mathcal{T}$ be the collection of subsets of X. Then, $\mathcal{T}$ is the discrete topology of X.

Example 1.3 (cofinite topology). Let X be any set and

$$\mathcal{T} = \{X \setminus U : U \subseteq X \text{ is finite}\} \cup \{\emptyset\}.$$

This is known as the cofinite topology of X since we are considering all sets $X \setminus U$ such that its complement in X, which is U, is finite.

Example 1.4 (cocountable topology). Let X be any set and

$$\mathcal{T} = \{U : U \text{ is open if } U = \emptyset \quad \text{or} \quad X \setminus U \text{ is countable.}\}$$

This is known as the cocountable topology of X.

Example 1.5 (cocountable topology on $\mathbb{R}$). An example of a cocountable topology can be constructed using any uncountable set X. Take $X = \mathbb{R}$. In the cocountable topology on $\mathbb{R}$, the open sets are defined as follows: $\emptyset$ is an open set and any set $U \subseteq \mathbb{R}$ such that $\mathbb{R} \setminus U$, which is countable, is also an open set. In other words, the latter means that U is open if its complement in $\mathbb{R}$ is countable.

By definition, $\emptyset$ is open. $\mathbb{R}$ is open in $\mathbb{R}$ since $\mathbb{R} \setminus \mathbb{R} = \emptyset$, which is countable. Any open interval, say $(0,1)$, is not open in the cocountable topology since its complement, $\mathbb{R} \setminus (0,1) = (-\infty, 0] \cup [1, \infty)$, is uncountable. However, a set like $\mathbb{R} \setminus \mathbb{Q}$ (the irrational numbers) is open because its complement $\mathbb{Q}$ is countable.

We conclude that the cocountable topology on $\mathbb{R}$ consists of $\emptyset, \mathbb{R}$, and all subsets of R whose complements are countable.

Example 1.6. Let $X = \mathbb{R}$ and

$$\mathcal{T} = \{(-\alpha, \alpha) : \alpha > 0\} \cup \{X, \emptyset\}.$$

Then, $(X, \mathcal{T})$ is a topological space. Note that the infinite intersection of $\mathcal{T}$ does not belong to $\mathcal{T}$, i.e.

$$\bigcap_{n=1}^{\infty} \left(-1 - \frac{1}{n}, 1 + \frac{1}{n}\right) = [-1, 1] \notin \mathcal{T}.$$

Example 1.7. Let $X = \{a, b, c\}$ and

$$\mathcal{T} = \{\{a\}, \{a, b\}, \emptyset, X\}.$$

Then, $(X, \mathcal{T})$ is a topological space.

1.2. *Describing Topologies*

Definition 1.2 (basis for topology)**.** Let X be a set. A basis for a topology of X is a collection $\mathcal{B}$ of a subset of X such that the following hold:

- **(i)** $\mathcal{B}$ covers X, i.e. for all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$
- **(ii)** for all $x \in X$ and $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \cap B_2$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$

The collection of sets $\mathcal{T}$ generated by $\mathcal{B}$ is the set

$$\mathcal{T} = \{U \subseteq X : \forall x \in U, \exists B \in \mathcal{B} \text{ such that } x \in B \subseteq U\}.$$

We say that $\mathcal{B}$ is a basis for $\mathcal{T}$.

Lemma 1.1. If $B_1, \ldots, B_n \in \mathcal{B}$ and $x \in B_1 \cap \ldots \cap B_n$, then there exists $B \in \mathcal{B}$ such that

$$x \in B \subseteq \bigcap_{i=1}^{n} B_i.$$

Proof. We proceed with induction on n. The base case is $n = 1$, which is obvious by taking $B = B_1$. Suppose $B_1, \ldots, B_n \in \mathcal{B}$ and $x \in B_1 \cap \ldots \cap B_n$. Then, we have $x \in B_1 \cap \ldots \cap B_{n-1}$. By the induction hypothesis, there exists $B' \in \mathcal{B}$ such that

$$x \in B' \subseteq \bigcap_{i=1}^{n-1} B_i.$$

Since $B', B_n \in \mathcal{B}$ and $x \in B' \cap B_n$, then by Definition 1.2, there exists $B \in \mathcal{B}$ such that

$$x \in B \subseteq B' \cap B_n \subseteq \bigcap_{i=1}^{n} B_i$$

and the proof is complete. □

Proposition 1.1. $\mathcal{T}$ is a topology.

Proof. By Definition 1.2, it is clear that $\emptyset, X \in \mathcal{T}$.

Suppose $\{U_\alpha\}_{\alpha\in I} \subseteq T$. Given

$$x \in \bigcup_{\alpha\in I} U_\alpha, \quad \text{there exists at least one } \alpha_x \in I \text{ such that } x \in U_{\alpha_x}.$$

By the definition of $\mathcal{T}$, there exists $B \in \mathcal{B}$ such that

$$x \in B \subseteq U_{\alpha_x} \subseteq \bigcup_{\alpha\in I} U_\alpha \quad \text{which implies} \quad \bigcup_{\alpha\in I} U_\alpha \in \mathcal{T}.$$

Suppose $U_1, \ldots, U_n \in \mathcal{T}$. Given

$$x \in \bigcap_{i=1}^{n} U_i \quad \text{then} \quad x \in U_i \text{ for all } 1 \le i \le n.$$

This shows that for all $1 \le i \le n$, there exists $B_i \in \mathcal{B}$ such that $x \in B_i \subseteq U_i$. Then, by Lemma 1.1, this shows that there exists $B \in \mathcal{B}$ such that

$$x \in B \subseteq \bigcap_{i=1}^{n} \subseteq \bigcap_{i=1}^{n} U_i \quad \text{so} \quad \bigcap_{i=1}^{n} U_i \in \mathcal{T}.$$

□

Example 1.8. Let X be a set. Show that if $\mathcal{B}$ is a basis of topology $\mathcal{T}$, then $\mathcal{T}$ equals the collection of all unions of elements in $\mathcal{B}$.

Solution. Suppose $\mathcal{B}$ is a basis of $\mathcal{T}$. Then,

$$\mathcal{T} = \{U \subseteq X : \forall x \in U, \exists B \in \mathcal{B} \text{ such that } x \in B \subseteq U\}.$$

We wish to prove that

$$U = \bigcup_{x\in U} B_x.$$

Given $B \in \mathcal{B}$, for any $x \in B$, we have $x \in B \subseteq B$, so B is an open set. Hence, $\mathcal{B} \subseteq \mathcal{T}$. Since $\mathcal{T}$ is a topology, it is closed under arbitrary unions, so

$$\bigcup_{B\in\mathcal{B}} B \in \mathcal{T}.$$

Conversely, let $U \in \mathcal{T}$. Since U is an open set, given $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B_x \subseteq U$. So,

$$\bigcup_{x\in U} B_x \subseteq U.$$

Since

$$B_x \subseteq \bigcup_{x\in U} B_x \text{ for all } x \in U \quad \text{then} \quad U \subseteq \bigcup_{x\in U} B_x.$$

□

Example 1.9 (open balls). Let $X=\mathbb{R}^n$. Then for any $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$ and $r>0$, define

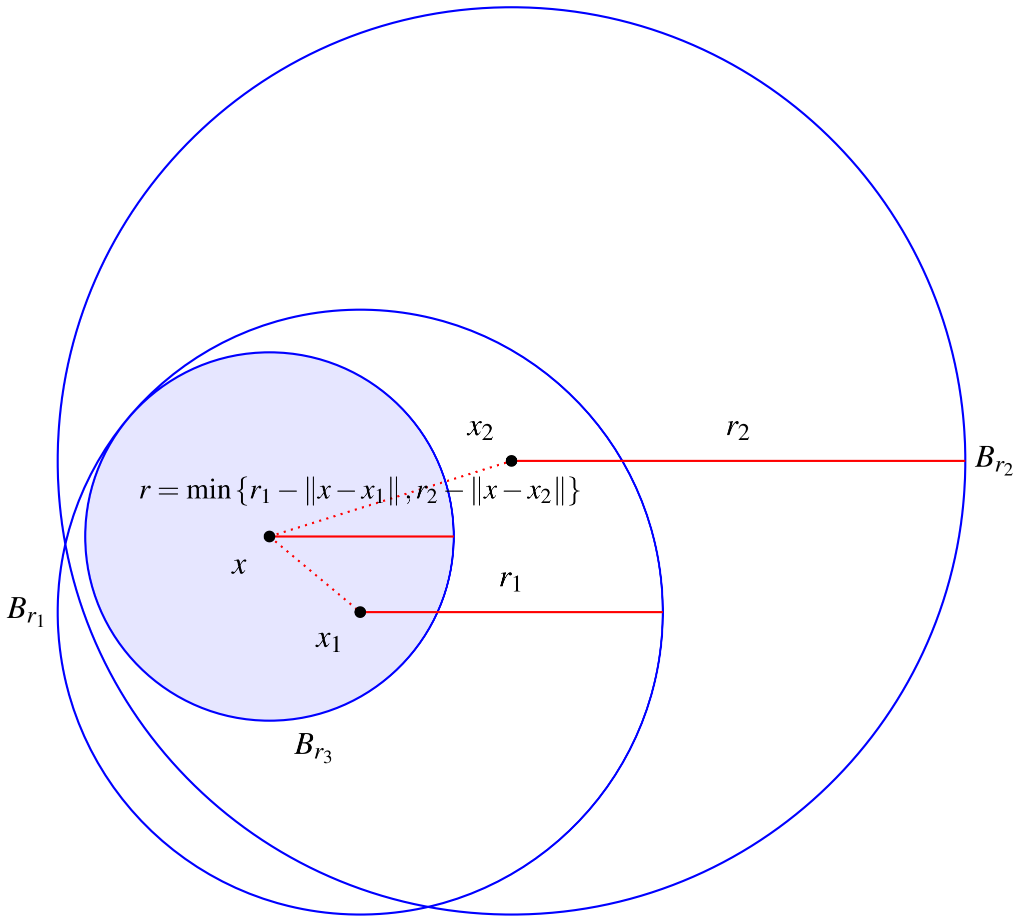
$$B_r(x)=\{y=(y_1,\ldots,y_n)\in\mathbb{R}^n:\sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2}<r\}.$$

Prove that $\mathcal{B}=\{B_r(x):x\in\mathbb{R}^n,r\in\mathbb{R}^+\}$ is a basis on $\mathbb{R}^n$.

Here, the topology on $\mathbb{R}^n$ generated by $\mathcal{B}$ is the standard topology.

Solution. This problem essentially states that open balls in $\mathbb{R}^n$ form a basis for the topology on a metric space. Recall that a basis for a topology of a set X is a collection $\mathcal{B}\subseteq X$ such that $\mathcal{B}$ covers X (i.e. $\forall x\in X,\exists B\in\mathcal{B}$ such that $x\in B$) and $\forall x\in X$ and $B_{r_1},B_{r_2}\in\mathcal{B}$ such that $x\in B_{r_1}\cap B_{r_2}$, there exists $B_{r_3}\in\mathcal{B}$ such that $x\in B_{r_3}\subseteq B_{r_1}\cap B_{r_2}$.

Refer to the following diagram:



Suppose $x\in\mathbb{R}^n$. Then, we can choose the r-ball centred at x, where $r>0$ to be in our basis such that $x\in B$. Since x is arbitrary, then $\mathcal{B}$ covers $\mathbb{R}^n$. To prove the second property, suppose for all $x_1,x_2\in X$, there exist $B_{r_1}(x_1),B_{r_2}(x_2)$ such that $x\in B_{r_1}\cap B_{r_2}$. Then, we can find $B_{r_3}\in\mathcal{B}$ such that $x\in B_{r_3}$. In particular, we can choose $x\in\mathbb{R}^n$ can be arbitrarily chosen and $r=\min\{r_1-\|x-x_1\|,r_2-\|x-x_2\|\}$.

□

Definition 1.3 (finer and coarser topologies)**.** Let X be a set and $\mathcal{T},\mathcal{T}'$ be topologies on X. We say that

$$T \text{ is finer than } T' \quad \text{if and only if} \quad T' \subseteq T.$$

Conversely, T' is coarser than T.

Example 1.10. Let $X=\mathbb{R}$ and define $\mathcal{T},\mathcal{T}'$ to be the following:

$$T=\left\{(-\alpha,\alpha):\alpha\in\mathbb{R}^+\right\}\cup\{\emptyset,\mathbb{R}\} \quad \text{and} \quad T' = \text{standard topology on } \mathbb{R}.$$

By the standard topology, we mean that $\mathcal{T}'$ is generated by the open intervals (a,b), where $a<b$ and $a,b\in\mathbb{R}$, meaning that it includes any arbitrary open interval as well as unions of such intervals.

As such, $\mathcal{T}'$ is finer than $\mathcal{T}$ and conversely, $\mathcal{T}$ is coarser than $\mathcal{T}'$.

Remark 1.1. The topology generated by a basis $\mathcal{B}$ is the coarsest topology that contains $\mathcal{B}$.

Proposition 1.2. Let

$$\mathcal{B},\mathcal{B}' \quad \text{be based for the topologies} \quad \mathcal{T},\mathcal{T}' \quad \text{respectively on } X..$$

Then, the following are equivalent:

(1) $\mathcal{T}'$ is finer than $\mathcal{T}$

(2) for all $B\in\mathcal{B}$, for all $x\in B$, there exists $B'\in\mathcal{B}'$ such that $x\in B'\subseteq B$

Proof. We first prove **(1)** implies **(2)**. Suppose $B\in\mathcal{B}$ is arbitrary. Then, because $\mathcal{T}'$ is finer than $\mathcal{T}$, then $\mathcal{T}\subseteq\mathcal{T}'$, so it follows that $B\in\mathcal{T}\subseteq\mathcal{T}'$. Recall that $\mathcal{B}'$ is a basis for $\mathcal{T}'$, so for every $x\in B$, there exists $B'\in\mathcal{B}'$ such that $x\in B'\subseteq B$.

We then prove **(2)** implies **(1)**. Suppose $U\in\mathcal{T}$ and $x\in U$. Since $\mathcal{B}$ is a basis for $\mathcal{T}$, then there exists $B\in\mathcal{B}$ such that $x\in B\subseteq U$. By assumption, there exists $B'\in\mathcal{B}'$ such that $x\in B'\subseteq B\subseteq U$. Since x was chosen arbitrarily, then $U\in\mathcal{T}'$ so $\mathcal{T}\subseteq\mathcal{T}'$. □

Example 1.11 (arithmetic progression topology)**.** An arithmetic sequence on $\mathbb{Z}$ is a set of the form

$$S(a,b):=\{an+b:n\in\mathbb{Z}\},$$

where $a,b\in\mathbb{N}$, $a>0$. Define on $X=\mathbb{Z}$ the collection of sets $\mathcal{T}$ in which the non-empty open sets are the unions of arithmetic sequences. Show that $\mathcal{T}$ is a topology on X.

Solution. We need to show the following:

(i) $\emptyset,X\in\mathcal{T}$

(ii) for $Y_1, \ldots \in \mathcal{T}$, we have

$$\bigcup_{n=1}^{\infty} Y_n \in \mathcal{T} \quad \text{and} \quad \bigcap_{n=1}^{N} Y_n \in \mathcal{T}.$$

By definition, $\emptyset \in \mathcal{T}$ since the empty set is the union of no arithmetic sequences; $X = \mathbb{Z} \in \mathcal{T}$ since we can set $a = 1$ and $b = 0$. Hence, **(i)** is satisfied. Next, let $Y_1, \ldots, Y_N \in \mathcal{T}$. That is to say, $Y_1, \ldots, Y_N$ are the non-empty open sets, and they are also the unions of arithmetic sequences, i.e.

$$Y_1 = \bigcup_{i_1 \in I_1} S(a_{i_1}, b_{i_1}) \quad Y_2 = \bigcup_{i_2 \in I_2} S(a_{i_2}, b_{i_2}) \quad \text{so in general,} \quad Y_n = \bigcup_{i_n \in I_n} S(a_{i_n}, b_{i_n}).$$

Hence,

$$Y_1 \cap \ldots \cap Y_N = \bigcup_{i_k \in I_k \forall 1 \le k \le N} S(a_{i_1}, b_{i_1}) \cap \ldots \cap S(a_{i_N}, b_{i_N})$$

The intersection $S(a_{i_1}, b_{i_1}) \cap \ldots \cap S(a_{i_N}, b_{i_N})$ is either empty or another arithmetic sequence. To see why, suppose $S(a_{i_1}, b_{i_1}) \cap \ldots \cap S(a_{i_N}, b_{i_N}) \neq \emptyset$. Then, use strong induction to conclude that $\mathcal{T}$ is closed under finite intersection (lazy to fill in the details for now). Hence, **(ii)** is satisfied. Also,

$$\bigcup_{n=1}^{\infty} Y_n = \bigcup_{n=1}^{\infty} \bigcup_{i_k \in I_k \forall k \in \mathbb{N}} S(a_{i_1}, b_{i_1}) \cup \ldots \cup S(a_{i_N}, b_{i_N})$$

which is the union of arithmetic sequences, so **(iii)** is satisfied. □

The interested reader can read up the Furstenberg topology and Golomb topology, which are used to prove the infinitude of primes, as well as Dirichlet's theorem on the infinitude of primes respectively.

Example 1.12 (MA3209 AY24/25 Sem 1 Homework 2)**.** Let $\mathcal{C}$ be a collection of subsets of X. Assume that $\emptyset, X \in \mathcal{C}$ and that finite unions and arbitrary intersections of sets in $\mathcal{C}$ are in $\mathcal{C}$. Show that

$$\mathcal{T} = \{X \setminus C : C \in \mathcal{C}\} \quad \text{is a topology on } X$$

and the collection of closed sets in this topology is $\mathcal{C}$.

Solution. First, we note that $\emptyset, X \in \mathcal{T}$. Next, suppose $U_1, \ldots \in \mathcal{C}$. Then,

$$\bigcup_{n=1}^{N} U_n \in \mathcal{C} \quad \text{and} \quad \bigcap_{n=1}^{\infty} U_n \in \mathcal{C}.$$

Suppose $X \setminus U_1, \ldots \in \mathcal{T}$. Then,

$$\bigcup_{n=1}^{\infty} (X \setminus U_n) = X \setminus \bigcap_{n=1}^{\infty} U_n \in X \setminus \mathcal{C} = \mathcal{T}.$$

Similarly,

$$\bigcap_{n=1}^{N} (X \setminus U_n) = X \setminus \bigcup_{n=1}^{N} U_n \in X \setminus \mathcal{C} = \mathcal{T}.$$

This shows that $\mathcal{T}$ is closed under arbitrary union and finite intersection. □

Example 1.13 (MA3209 AY24/25 Sem 1 Homework 1)**.** Order the following topologies on $X = [0,1]$ according to finer/coarser:

(i) the trivial topology;

(ii) the discrete topology;

(iii) Euclidean topology;

(iv) the co-finite topology;

Compare the cocountable topology with the topologies that is comparable in the above list.

Solution. We recall the following: the trivial topology $\mathcal{T}$ can be regarded as $\{\emptyset, X\}$; the discrete topology can be regarded as the collection of subsets of X; the Euclidean topology is the standard topology inherited from $\mathbb{R}$; the co-finite topology refers to the collection of subsets $\{\emptyset\} \cup X \backslash U$, where $U \subseteq X$ is finite.

Note that $[0,1]$ is an uncountable set and

$$[0,1] \quad \text{is equinumerous to} \quad \mathbb{R}$$

so the collection of subsets of $[0,1]$ is equinumerous to $\mathcal{P}(\mathbb{R})$. Hence, it is clear that the discrete topology is the finest, whereas the trivial topology is the coarsest.

We then order the Euclidean and co-finite topologies with respect to the other two. Obviously, both topologies are finer than the trivial topology but coarser than the discrete topology. We claim that the Euclidean topology is finer than the co-finite topology. Consider some element in the co-finite topology, i.e. U is contained in the co-finite topology such that $[0,1] \backslash U$ is finite. Consider a subset

$$(a,b) \subseteq [0,1] \quad \text{where } 0 \leq a < b \leq 1.$$

Then, the complement of (a,b) in $[0,1]$ is $[0,a) \cup (b,1]$, which is uncountable. Hence, (a,b) is not open in the cofinite topology since its complement is not finite. Hence, for the first part of the problem, our comparison is as follows:

$$\text{trivial topology} \subseteq \text{co-finite topology} \subseteq \text{Euclidean topology} \subseteq \text{discrete topology}$$

For the second part of the problem, our comparison is as follows:

$$\text{trivial topology} \subseteq \text{co-finite topology} \subseteq \text{cocountable topology} \subseteq \text{discrete topology}$$

As such, it suffices to prove the following results:

- the Euclidean topology $\mathcal{T}_1$ is not comparable with the cocountable topology $\mathcal{T}_2$
- the cocountable topology is finer than the co-finite topology

For the first result, suppose we have some element of $\mathcal{T}_1$, say open intervals (a,b), where $0 \leq a < b \leq 1$. Similar to an argument made earlier when we were comparing the co-finite topology with the Euclidean topology, we have $\mathcal{T}_1 \subsetneq \mathcal{T}_2$. Conversely, take some element of $\mathcal{T}_2$, say $\mathbb{R} \backslash \{1/n : n \in \mathbb{N}\}$.

Note that its complement in $\mathbb{R}$, which is $\{1/n : n \in \mathbb{N}\}$, is countable. Then, the set $\mathbb{R} \setminus \{1/n : n \in \mathbb{N}\}$ is not open in $\mathcal{T}_1$ as $0 \in \mathcal{T}_1$ but all neighbourhoods of intersect an element of $\mathcal{T}_1^c$.

Lastly, we show that the cocountable topology is finer than the co-finite topology. Consider some element U of the co-finite topology. Then, $X \setminus U$ is finite, which is also countable. It follows that U is also an element of the cocountable topology. □

Definition 1.4 (subbasis). A subbasis $\mathcal{S}$ of a set X is a collection of subsets of X whose union equals X. The topology generated by $\mathcal{S}$ is a collection $\mathcal{T}$ of all unions of finite intersection of sets in $\mathcal{S}$.

Proposition 1.3. The topology generated by a subbasis $\mathcal{S}$ is a topology.

Proof. Here, we define a subbasis $\mathcal{S}$ of a set X to be a collection of subsets of X whose union equals X. The topology generated by $\mathcal{S}$ is a collection $\mathcal{T}$ of all unions of finite intersections of sets in $\mathcal{S}$.

Since $\mathcal{S}$ is a subbasis of X, then

$$\bigcup \mathcal{S} = X \quad \text{so } X \text{ can be written as a union of elements in } \mathcal{S}$$

As such, $X \in \mathcal{T}$. The empty set $\emptyset$ can also be written as an empty union which is also an element of $\mathcal{T}$. So, $\emptyset, X \in \mathcal{T}$.

We then prove that $\mathcal{T}$ is closed under arbitrary unions. Suppose $\{U_\alpha\}_{\alpha \in I}$ is a collection of sets in $\mathcal{T}$. Then, each U_α is a union of finite intersections of sets in $\mathcal{S}$. So,

$$\bigcup_{\alpha \in I} U_\alpha \quad \text{is also a union of finite intersections of sets in } \mathcal{S} \quad \text{which implies} \quad \bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}.$$

Lastly, we prove that $\mathcal{T}$ is closed under finite intersections. Suppose $U_1, \ldots, U_n \in \mathcal{T}$. Then, each U_i can be written as a union of finite intersections of sets in $\mathcal{S}$. So, $U_1 \cap \ldots \cap U_n$ can also be written as a union of finite intersections of sets in $\mathcal{S}$. □

1.3. *Metric Spaces*

Definition 1.5 (metric). A metric/distance on a set X is a function $d : X \times X \to \mathbb{R}$ such that the following conditions are satisfied:

- **(i) Non-negativity:** $d(x,y) \geq 0$ for all $x, y \in X$
- **(ii) Positive definiteness:** $d(x,y) = 0$ if and only if $x = y$
- **(iii) Symmetry:** $d(x,y) = d(y,x)$ for all $x, y \in X$
- **(iv) Triangle inequality:** $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$

If the mentioned conditions are satisfied, then (X,d) is a metric space.

Definition 1.6 (pseudometric). If **(i)**, **(iii)** and **(iv)** of Definition 1.5 are satisfied and $d(x,x) = 0$ for all $x \in X$, then the corresponding function d will be a pseudometric.

Example 1.14. Let $\mathcal{F}([0,1])$ be the space of real-valued functions defined on $[0,1]$. Then, for any

$$f, g \in \mathcal{F}([0,1]), \quad \text{we define} \quad d(f,g) = |f(0) - g(0)|.$$

Here, d is a pseudometric on $\mathcal{F}([0,1])$.

Example 1.15 (pseudometric but not metric). Note that every metric space is also a pseudometric space. However, the converse is not true. For example, let X be the set such that $|X| > 1$. Consider the function

$$d : X \times X \to \mathbb{R} \quad \text{where} \quad d(x,y) = 0.$$

Clearly, d is a pseudometric but it is not a metric as we can have distinct $x, y \in X$ but $d(x,y) = 0$.

Definition 1.7 (quasimetric). If **(i)**, **(ii)** and **(iv)** of Definition 1.5 hold, the corresponding function d will be a quasimetric.

Example 1.16. We have

$$d(x,y) = \begin{cases} x - y & \text{if } x \geq y; \\ 1 & \text{otherwise} \end{cases} \quad \text{being a quasimetric on reals.}$$

Example 1.17 (discrete metric). Suppose X is a set. Define

$$d : X \times X \to \mathbb{R} \quad \text{where} \quad d(x,y) = \begin{cases} 1 & \text{if } x \neq y; \\ 0 & \text{if } x = y. \end{cases}$$

This is called the discrete metric.

Definition 1.8 (norm). Let F be a field (for example, $\mathbb{R}$ or $\mathbb{C}$). A norm on an F-vector space V is a function

$$\|\cdot\| : V \to \mathbb{R}$$

which satisfies the following:

(i) Non-negativity: $\|x\| \geq 0$ for all $x \in V$

(ii) Positive definiteness: $\|x\| = 0$ if and only if $x = 0$

(iii) Absolute homogeneity: $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in F$ and $x \in V$

(iv) Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$

Here are some examples of norms.

Example 1.18 (Euclidean norm). Let $V = \mathbb{R}^n$. Define the L^2-norm or the Euclidean norm to be the following:

$$\|x\|_2 = \sqrt{x_1^2 + \ldots + x_n^2} \quad \text{for all } x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

Example 1.19 (Hermitian norm). Let $V = \mathbb{C}^n$. Similar to Example 1.18 on the Euclidean norm, we define

$$\|x\|_2 = \sqrt{x_1\bar{x}_1 + \ldots + x_n\bar{x}_n} \quad \text{for all } x = (x_1, \ldots, x_n) \in \mathbb{C}^n.$$

This is said to be the Hermitian norm. In fact, the use of the word *Hermitian* is not surprising — recall from MA2101 that a matrix $\mathbf{A} = (a_{ij})$ is Hermitian if and only if it is equal to its conjugate transpose, i.e. $a_{ij} = \overline{a_{ji}}$.

Example 1.20 (p-norm). Let $V = F^n$, where F is an arbitrary field. Suppose $p \geq 1$. Then, define the p-norm to be the following:

$$\|x\|_p = (|x_1|^p + \ldots + |x_n|^p)^{1/p} \quad \text{for all } x \in F^n$$

Example 1.21 (infinity norm/supremum norm). Let $V = F^n$. Define the supremum norm to be the following:

$$\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\} \quad \text{for all } x \in F^n$$

Remark 1.2. The metric induced by the Euclidean norm is known as the Euclidean metric. The same relationship can be said for the other three metrics and norms that we discussed.

Proposition 1.4. Every norm $\|\cdot\|$ on V induces a metric d on V by

$$d(x, y) = \|x - y\| \quad \text{for all } x, y \in V.$$

The proof of Proposition 1.4 is trivial.

Definition 1.9 (Hausdorff metric). Let X be the space of all closed subsets of the Euclidean space $\mathbb{R}^n$. Let

$$B_\varepsilon(A) = \bigcup_{a \in A} B_\varepsilon(a) \quad \text{be an } \varepsilon\text{-neighbourhood of the set } A.$$

Define the Hausdorff metric, $d_H(A, B)$ to be the following:

$$d_H(A, B) = \inf\{\varepsilon > 0 : A \subseteq B_\varepsilon(B) \text{ and } B \subseteq B_\varepsilon(A)\}$$

Example 1.22 (MA3209 AY24/25 Sem 1 Homework 1). Let X be the space of all closed subsets of the Euclidean space $\mathbb{R}^n$. Let

$$B_\varepsilon(A) = \bigcup_{a \in A} B_\varepsilon(a) \quad \text{be an } \varepsilon\text{-neighbourhood of the set } A.$$

Show that the so-called Hausdorff metric

$$d_H(A,B) = \inf\{\varepsilon > 0 : A \subseteq B_\varepsilon(B) \text{ and } B \subseteq B_\varepsilon(A)\}$$

is indeed a metric on X.

Solution. By definition, $A \subseteq B_\varepsilon(B)$ and $B \subseteq B_\varepsilon(A)$ imply that A is contained in an ε-neighbourhood of B and B is contained in an ε-neighbourhood of A. We wish to prove that d_H satisfies non-negativity, symmetry, and the triangle inequality.

- **Non-negativity:** Since $d_H(A,B)$ is derived from distances between points in A and B, $d_H(A,B) \geq 0$.
- **Symmetry:** We have

$$\begin{aligned} d_H(B,A) &= \inf\{\varepsilon > 0 : B \subseteq B_\varepsilon(A) \text{ and } A \subseteq B_\varepsilon(B)\} \\ &= \inf\{\varepsilon > 0 : A \subseteq B_\varepsilon(B) \text{ and } B \subseteq B_\varepsilon(A)\} = d_H(A,B) \end{aligned}$$

- **Triangle inequality:** By definition, we have

$$\begin{aligned} d_H(A,B) &= \inf\{\varepsilon > 0 : A \subseteq B_\varepsilon(B) \text{ and } B \subseteq B_\varepsilon(A)\} \\ d_H(B,C) &= \inf\left\{\varepsilon' > 0 : B \subseteq B_{\varepsilon'}(C) \text{ and } C \subseteq B_{\varepsilon'}(B)\right\} \end{aligned}$$

  We wish to prove that

$$d_H(A,C) = \inf\left\{\varepsilon'' > 0 : A \subseteq B_{\varepsilon''}(C) \text{ and } C \subseteq B_{\varepsilon''}(A)\right\} \leq d_H(A,B) + d_H(B,C).$$

  Suppose $x \in A$. Then, $x \in B_\varepsilon(B)$. So, $x \in B_\varepsilon(b)$ for some open ball of radius ε centred at b, i.e. there exists $b \in B$ such that $d(x,b) < \varepsilon$. Since $b \in B \subseteq B_{\varepsilon'}(C)$, then there exists $c \in C$ such that $d(b,c) < \varepsilon'$. Since the distance function is a metric, then by the triangle inequality, it follows that

$$d(x,c) \leq d(x,b) + d(b,c) < \varepsilon + \varepsilon' \quad \text{which implies} \quad A \subseteq B_{\varepsilon+\varepsilon'}(C)$$

  In a similar fashion, one can prove that $C \subseteq B_{\varepsilon+\varepsilon'}(A)$. Choosing $\varepsilon'' = \varepsilon + \varepsilon'$ and taking infimimums yields the desired inequality. □

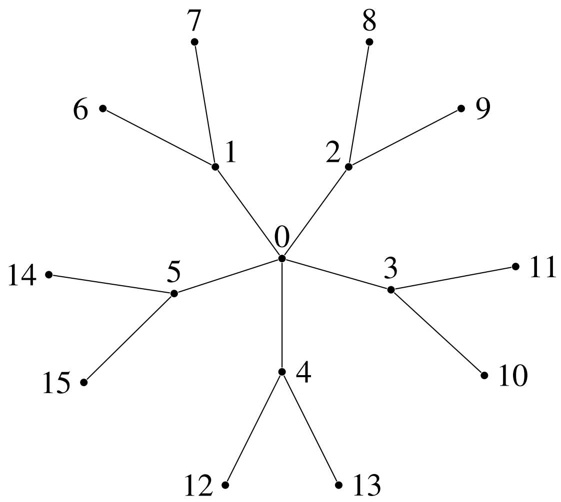
Example 1.23. Is the Hausdorff metric d_H also a metric on the space of all subsets of $\mathbb{R}^n$? Justify your answer.

Solution. No. Let X be the space of all subsets of $\mathbb{R}$. Suppose $A = \{1\}$ and $B = \mathbb{R}$.

$$d_H(A,B) = \inf\{\varepsilon > 0 : A \subseteq B_\varepsilon(B) \text{ and } B \subseteq B_\varepsilon(A)\} = \infty.$$

In general, we can let X be some Euclidean space $\mathbb{R}^n$ and let $A, B \subseteq X$ be unbounded sets. □

Example 1.24 (Graph Theory)**.** Let X denote the set of vertices of a graph G and $d(x,y)$ denote the length of the shortest path between x and y. Then, d is indeed a metric.



For example, we can consider our graph G to be the tree above. Set $X = \{0,1,2,\ldots,15\}$. For example, the triangle inequality is satisfied because $d(5,9) \leq d(5,4) + d(4,9)$.

Example 1.25 (p-adic metric)**.** Let $X = \mathbb{Q}$ and p be a prime number. Then,

$$\text{for all } x \in \mathbb{Q}\setminus\{0\}, \quad \text{there exists} \quad \text{a unique } k \in \mathbb{Z} \text{ such that } x = \frac{p^k r}{s},$$

where $r, s \in \mathbb{Z}$ which are not divisible by p.

Then, define

$$|x|_p = \begin{cases} p^{-k} & \text{if } x = p^k r/s \text{ as above;} \\ 0 & \text{if } x = 0. \end{cases}$$

Define $d(x,y) = |x-y|_p$, which is called the p-adic metric.

Proposition 1.5 (p-adic metric)**.** $|\cdot|_p$ is a norm on the rational numbers and $d(x,y)$ is a metric.

Remark 1.3 (ultrametric)**.** In fact, the p-adic metric satisfies the strong triangle inequality, i.e.

$$d(x,z) \leq \max\{d(x,y), d(y,z)\} \quad \text{for all } x,y,z \in \mathbb{Q}.$$

Such metrics are called ultrametrics.

We now prove Proposition 1.5.

Proof. To show that $|\cdot|_p$ is a norm, we need to prove that it is non-negative, absolutely homogeneous, and satisfies the triangle inequality. We will only prove that it satisfies the triangle inequality. Note that

$$|x|_p = \begin{cases} p^{-k} & \text{if } x = p^k r/s; \\ 0 & \text{if } x = 0 \end{cases} \quad \text{and} \quad |y|_p = \begin{cases} p^{-m} & \text{if } y = p^m r'/s'; \\ 0 & \text{if } y = 0 \end{cases}$$

If $x = 0$ or $y = 0$, then the proof is quite trivial. Suppose for all $x, y \in \mathbb{Q} \setminus \{0\}$, there exist unique $k, m \in \mathbb{Z}$, where $k \leq m$, such that $x = p^k r/s$ and $y = p^m r'/s'$. Hence,

$$x + y = \frac{p^k r}{s} + \frac{p^m r'}{s'} = \frac{p^k rs' + p^m r's}{ss'} = p^k \left(\frac{rs' + p^{m-k} r's}{ss'} \right)$$

so, $|x+y|_p = p^{-k} \leq \max\left\{|x|_p, |y|_p\right\}$. Here, we proved the strong triangle inequality.

We then prove that $d(x,y)$ is a metric. Again, we only justify that the triangle inequality holds as the other properties are rather trivial. By definition, we have $d(x,y) = |x-y|_p$ and $d(y,z) = |y-z|_p$, or rather,

$$|x-y|_p = \begin{cases} p^{-k_1} & \text{if } x - y = p^{k_1} r_1/s_1; \\ 0 & \text{if } x - y = 0 \end{cases} \quad \text{and} \quad |y-z|_p = \begin{cases} p^{-k_2} & \text{if } y - z = p^{k_2} r_2/s_2; \\ 0 & \text{if } y - z = 0. \end{cases}$$

Say $k_1 \leq k_2$. We then use the trick that $x - z = (x-y) + (y-z)$ so deduce that $|x-z|_p = p^{-k_1}$ and so this is bounded above by $|x-y|_p + |y-z|_p$ (or we can use the strong triangle inequality to conclude). □

Definition 1.10 (Minkowski distance). Let $V = \mathbb{R}^n$ and $p \geq 1$. We define for all $x, y \in V$,

$$d_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p} \quad \text{to be the Minkowski distance.}$$

Example 1.26 (MA3209 AY24/25 Sem 1 Homework 1). Based on Definition 1.10

$$d_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p},$$

Prove that d_p is really a metric for every $p \geq 1$.

Solution. We will only verify that d_p satisfies the triangle inequality. By definition,

$$d_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p} \quad \text{and} \quad d_p(y,z) = \left(\sum_{i=1}^{n} |y_i - z_i|^p \right)^{1/p}.$$

Then, $d_p(x,z) \leq d_p(x,y) + d_p(y,z)$ by Minkowski's inequality. In fact, Minkowski's inequality can be deduced using Hölder's inequality. □

Example 1.27 (MA3209 AY24/25 Sem 1 Homework 1). Let

$$\ell_p = \left\{ \{x_n\}_{n \in \mathbb{N}} : x_n \in \mathbb{C} \text{ and } \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}.$$

This is related to the p-norm in Functional Analysis. Prove that $\ell_p \subseteq \ell_q$ for all $1 \leq p \leq q$.

Solution. Note that $p,q \in \mathbb{R}$ such that $1 \leq p \leq q$. Suppose $x_n \in \ell_p$. Then,

$$x_n \text{ is a sequence of complex numbers} \quad \text{such that} \quad \sum_{n=1}^{\infty} |x_n|^p \text{ is finite.}$$

So, the sum of $|x_n|^p$ is convergent, which implies that there exists $N \in \mathbb{N}$ such that for all $x \geq N$, we have $|x_N|^p < 1$. So, $|x_N|^q < 1$ since $q \geq p$. In fact, $|x_N|^q \leq |x_N|^p$. This implies that

$$\sum_{n \geq N} |x_n|^q \leq \sum_{n \geq N} |x_n|^p$$
$$\sum_{n=1}^{\infty} |x_n|^q \leq \sum_{n=1}^{N-1} |x_n|^q + \sum_{n \geq N} |x_n|^p \leq (N-1)\max\left(|x_1|^q, \ldots, |x_{N-1}|^q\right) + \sum_{n \geq N} |x_n|^p$$

which is the sum of a finite quantity and a finite sum. □

Definition 1.11 (distance and diameter)**.** Let A,B be non-empty subsets of a metric space (X,d).

(i) The distance between A and B is

$$d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}.$$

(ii) The diameter of $A \subseteq X$ is

$$\operatorname{diam}(A) = \sup\{d(x,y) : x,y \in A\}.$$

Definition 1.12 (bounded set)**.** A set $A \subseteq X$ is bounded if $\operatorname{diam}(A) < \infty$.

Definition 1.13 (metrizability)**.** The topology on X induced by a matrix d is the topology generated by $\mathcal{B}_d$. A topology on X is metrizable if there exists a metric on X that induces $\mathcal{T}$.

1.4. *Subspaces of Topological Spaces*

Definition 1.14 (subspace topology)**.** Let $(Y, \mathcal{T}_Y$ be a topological space and $X \subseteq Y$ be a subset. Then,

$$\mathcal{T}_X = \{U \cap X : U \in \mathcal{T}_Y\} \quad \text{is the subspace topology on } X.$$

Definition 1.15. Let $(Y, \mathcal{T}_Y)$ be a topological space, $X \subseteq Y$ and $\mathcal{T}_X$ is the subspace topology. Then, X is a subspace of Y with respect to this topology $\mathcal{T}_X$.

Definition 1.16 (restriction)**.** Let $A \subseteq (X,d)$, where (X,d) is a metric space. Then, the restriction of d to A is the metric

$$d_A(x,y) = d(x,y).$$

1.5. *Closed Sets, Closure and Limit Points*

Definition 1.17 (closed set)**.** Let $(X, \mathcal{T})$ be a topological space. A subset $A \subseteq X$ is closed if $X \backslash A \in \mathcal{T}$.

Example 1.28 (closed intervals in $\mathbb{R}$)**.** The closed interval $[a,b] \subseteq \mathbb{R}$ is closed with respect to the standard topology of $\mathbb{R}$.

Example 1.29. Let

$$X = [0,1] \cup (2,3) \subseteq \mathbb{R}.$$

Then, $[0,1]$ is both open and closed in X with respect to the subspace topology of X inherited from the standard topology of $\mathbb{R}$.

To see why $[0,1]$ is open in X, we need to find an open set $U \subseteq \mathbb{R}$ such that $[0,1] = U \cap X$. For example, we can choose $U = (-1,2)$. On the other hand, to see why $[0,1]$ is closed in X, we need to show that the complement of $[0,1]$ is open in X, i.e. $(2,3)$ is indeed open in X.

Definition 1.18 (interior, closure, boundary)**.** Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then,

(a) The interior of A is

$$\operatorname{int}(A) = A^\circ = \bigcup_{U \in \mathcal{T}, U \subseteq A} U.$$

(b) The closure of A is

$$\operatorname{cl}(A) = \overline{A} = \bigcap_{X \backslash G \in \mathcal{T}, A \subseteq G} G.$$

(c) The boundary of A is

$$\operatorname{bd}(A) = \partial A = \overline{A} \backslash A^\circ = \operatorname{cl}(A) \backslash \operatorname{int}(A).$$

Proposition 1.6. We have the following obvious results:

(1) $A^\circ \subseteq A \subseteq \overline{A}$

(2) $A^\circ = A$ if and only if A is open

(3) $\overline{A} = A$ if and only if A is closed

Example 1.30. Every point of an open interval $(a,b) \subseteq \mathbb{R}$ is an interior of (a,b).

Example 1.31. The empty set $\emptyset \subseteq \mathbb{R}$ is trivially an open set because there exists no point in $\emptyset$. Also, the real line $\mathbb{R}$ is itself an open set because every point in $\mathbb{R}$ is an interior point of $\mathbb{R}$.

Example 1.32. Let X be the real line space $\mathbb{R}$ and $A = (0,1] \subseteq \mathbb{R}$. Then, $\partial A = \{0,1\}$.

Example 1.33. Let $A = \{(x,y) \in \mathbb{R}^2 : 0 < x \leq 1, 0 < y \leq 1\}$. Prove

$$A^\circ = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1\},$$

$$\overline{A} = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\},$$

$$\partial A = ([0,1] \times \{0\}) \cup ([0,1] \times \{1\}) \cup (\{0\} \times [0,1]) \cup (\{1\} \times [0,1]).$$

Solution. To find A°, we consider the union of all open sets U that are contained in A. That is,

$$A^\circ = \bigcup_{U \in \mathcal{T}, U \subseteq A} U$$

so we note that the boundaries $x = 1$ and $y = 1$ cannot be part of any open set U. Suppose otherwise, then any ε-neighbourhood would contain points outside of A. So, the first result follows. To put it more rigorously, note that $(x,y) = (1/2, 1/2) \in A$. Then, there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \times (y-\varepsilon, y+\varepsilon) \in (0,1) \times (0,1)$, i.e. $\varepsilon = 1/2$. In fact, this is the largest open interval that is contained in A.

Then, recall that the closure

$$\overline{A} = \bigcap_{G \in \mathcal{T}, A \subseteq G} G.$$

$\overline{A}$ must include the boundary points $x = 0$ and $y = 0$ since any closed set containing A includes the boundary points. As we take the intersection of all such closed sets, the result follows. Again, to put it more rigorously, consider an arbitrary closed set G that contains A, i.e.

$$G = [a-c, a+c] \times [b-d, b+d] \supseteq (0,1] \times (0,1] = A$$

where

$$a - c \leq 0 < 1 \leq a + c \text{ and } b - d \leq 0 < 1 \leq b + d.$$

Taking the intersection over all such G, we can consider finding the smallest closed subset of G that contains A, which is $[0,1]$, i.e. set $a = c = b = d = 1/2$. Hence, the second result follows.

Suppose $v \in \partial A = \overline{A} - A^\circ$. Then, $v \in \overline{A}$ but $v \notin A^\circ$, i.e. $v \in [0,1] \times [0,1]$ but $v \notin (0,1) \times (0,1)$. Recall that for any sets A, B, C, D, the identity

$$(A \times B) \backslash (C \times D) = (A \times (B \backslash D)) \cup ((A \backslash C) \times B)$$

holds. Setting $A = B = [0,1]$ and $C = D = (0,1)$, the result follows. □

Definition 1.19 (limit point). Let X be a topological space and $A \subseteq X$. A point $x \in X$ is a limit point of A if every open $U \subseteq X$ containing x intersects $A \backslash \{x\}$.

Example 1.34. Let $A = \{0\} \cup (1,2) \subseteq \mathbb{R}$. This set has $[1,2]$ as its set of limit points with respect to the standard topology of $\mathbb{R}$. However, 0 is not a limit point of A. To see why, we can consider the open interval $(-1/2, 1/2)$, which does not intersect with $A \setminus \{0\}$.

In fact, any $x \in \mathbb{R} \setminus (\{0\} \cup [1,2])$ is not a limit point of A. To see why, the open set $(-\infty, 0) \cup (0,1) \cup (2,\infty)$, which is the union of open intervals, contains the above x but does not intersect $A \setminus \{x\} = A$.

As mentioned, every $x \in [1,2]$ is a limit point of A. To see why, note that for all $a,b \in \mathbb{R}$ such that $a < x < b$, note that

$$(a,b) \cap (1,2) \text{ is infinite} \quad \text{and} \quad (a,b) \cap (A \setminus \{x\}) \neq \emptyset.$$

Recall that for all $U \subseteq \mathbb{R}$, U is open and for all $x \in U$, there exist $a,b \in \mathbb{R}$ such that $x \in (a,b) \subseteq U$. This shows that $U \cap (A \setminus \{x\}) \neq \emptyset$.

On the other hand, if X is equipped with the discrete topology, then all subsets of X have no limit points.

Example 1.35. Find all the limit points of the following subsets of $\mathbb{R}$:

(a) $\left\{\frac{1}{m} + \frac{1}{n} : m,n = 1,2,\ldots\right\}$.

(b) $\left\{\frac{\sin n}{n} : n = 1,2,\ldots\right\}$.

Solution.

(a) When $m = n = 1$, we see that 2 is an element of the set. As $m \to \infty$ or $n \to \infty$, the sequence $1/m + 1/n$ is decreasing and tends to 0. So, we conclude that 0 is a limit point. In fact, it is the only limit point.

To put it more rigorously, using Definition 1.19, say $x \in X$ is a limit point of the sequence. Then, let $U \subseteq X$ be an open set containing x, i.e. there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq U$. Note that the open set contains x. Since $(x-\varepsilon, x+\varepsilon) \cap (A \setminus \{x\}) \neq \emptyset$, then there exists $m,n \in \mathbb{N}$ (without loss of generality, assume $m \geq n > 0$) such that

$$-\varepsilon = \frac{1}{m} + \frac{1}{n} \quad \text{or} \quad \varepsilon = \frac{1}{m} + \frac{1}{n}.$$

However, $\varepsilon > 0$ so we reject the first case. Working with the second case, we have

$$\varepsilon = \frac{m+n}{mn} \geq \frac{m+m}{mn} = \frac{2}{n} \quad \text{so we can choose} \quad n \geq \left\lceil \frac{2}{\varepsilon} \right\rceil.$$

As mentioned, only $x = 0$ is the limit point of the sequence.

(b) Note that $\sin n \leq 1$ for all $n \in \mathbb{N}$ so $\sin n/n \leq 1/n$, which implies we can consider the behaviour of the sequence $1/n$ as it tends to infinity. Since the latter converges to 0, we infer that 0 is a limit point of this subset; similar to the first set, we see that 0 is the only limit point. □

Proposition 1.7. Let X be a topological space and $A \subseteq X$. Then, the following hold:

(i) $x \in \overline{A}$ if and only if for all open $U \ni x$, we have $U \cap A \neq \emptyset$

(ii) If A' is the set of limit points of A, then $\overline{A} = A \cup A'$

Proof. We first prove **(i)**. Note that by contraposition, the statement is equivalent to $x \notin \overline{A}$ if and only if there exists open $U \ni x$ such that $U \cap A = \emptyset$. The forward direction follows by setting $U = X \backslash \overline{A}$. As for the reverse direction, we denote $G = X \backslash U$. Then, $G \subseteq X$ is closed and $A \subseteq G$, which shows that $\overline{A} \subseteq G$. As $x \notin G$, then $x \notin \overline{A}$.

We then prove **(ii)**. Recall that $A \subseteq \overline{A}$. **(i)** of the proposition implies $A' \subseteq \overline{A}$, so it follows that $A \cup A' \subseteq \overline{A}$. We then need to show that $\overline{A} \subseteq A \cup A'$. Suppose $x \in \overline{A}$ and that $x \notin A'$. This implies that there exists an open $U \ni x$ such that $U \cap (A \backslash \{x\}) = \emptyset$. By the contrapositive statement of **(i)**, we have $U \cap A = \emptyset$, so $x \in A$. □

Definition 1.20 (dense and nowhere dense sets)**.** Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then, we have the following:

(i) A is a dense set in X if $\overline{A} = X$

(ii) A is a nowhere dense set if $\left(\overline{A}\right)^\circ = \emptyset$. Note that $\left(\overline{A}\right)^\circ$ refers to the union of all open sets of X which are contained in $\overline{A}$.

We note that a subset A of a topological space X is dense in X if for any point $x \in X$, any neighbourhood of x contains at least one point from A, i.e. A has a non-empty intersection with every non-empty open subset of X. In other words, A is dense in X if the only closed subset of X containing A is X itself.

Example 1.36 ($\mathbb{Q}$ is dense in $\mathbb{R}$)**.** The set of rationals $\mathbb{Q}$ is dense in $\mathbb{R}$ with usual topology since in this topology, every real number is a limit point of $\mathbb{Q}$. Hence, $\overline{\mathbb{Q}} = \mathbb{R}$.

Example 1.37. Every one-point set in $\mathbb{R}$ is nowhere dense.

Example 1.38 ($\mathbb{R} \backslash \mathbb{Q}$ is dense in $\mathbb{R}$)**.** The set of all irrational numbers, denoted by $\mathbb{R} \backslash \mathbb{Q}$, is also dense in $\mathbb{R}$ with the usual topology since $\overline{\mathbb{R} \backslash \mathbb{Q}} = \mathbb{R}$.

Example 1.39. In the real line space $\mathbb{R}$, define the set A to be

$$A = \{x \in \mathbb{Q} : 0 < x < 1\},$$

which is not nowhere dense in $\mathbb{R}$ because $\overline{A} = [0,1]$ so the interior of this set is the open interval $(0,1)$. As this set is non-empty, the result follows.

Example 1.40. The real number space $\mathbb{R}$ with the usual topology has the rational numbers $\mathbb{Q}$ as a countable dense subset. This implies that the cardinality of a dense subset of a topological space may be strictly smaller than the cardinality of the space itself.

Example 1.41. Let

$$A=\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\right\}\subseteq\mathbb{R}\quad\text{so}\quad\overline{A}=\left\{0,1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\right\}.$$

Note that A is nowhere dense in $\mathbb{R}$ since $\overline{A}$ has no interior point, so the interior of this set is $\emptyset$.

Proposition 1.8. Let $(X,\mathcal{T})$ be a topological space and $A\subseteq X$. Then, the following hold:

(i) If A is open in $(X,\mathcal{T})$, then ∂A is nowhere dense in $(X,\mathcal{T})$

(ii) If A is closed in $(X,\mathcal{T})$, then ∂A is nowhere dense in $(X,\mathcal{T})$

(iii) If A is closed in X, then A is nowhere dense if and only if $X\backslash A$ is everywhere dense.

The following proposition (Proposition 1.9) characterises nowhere dense subsets of a metric space with the help of its open balls.

Proposition 1.9. Let (X,d) be a metric space. We say that $A\subseteq X$ is nowhere dense if and only if the following equivalent conditions hold:

(i) $\overline{A}$ does not contain any non-empty open ball

(ii) every non-empty open set has a non-empty open ball disjoint from $\overline{A}$

Definition 1.21 (convergence of a seuqence)**.** Let

$$(x_1,x_2,x_3,\ldots)=\{x_i\}_{i=1}^{\infty}\quad\text{be a sequence of points in a topological space } X.$$

We say that x_i converges to $x\in X$ if for any neighbourhood $U\ni x$, there exists $N>0$ such that $x_k\in U$ for all $k>N$. We write

$$x_i\to x\quad\text{and say that}\quad x\text{ is a limit of }\{x_i\}_{i=1}^{\infty}.$$

Remark 1.4. Take note of the following:

(i) x is a limit point of $\{x_1,\ldots\}$ does not imply that $x_i\to x$

(ii) $x_i\to x$ does not imply that x is a limit point of $\{x_1,\ldots\}$

Example 1.42. Note that

$$\{x_n\}_{n=1}^{\infty}=\left\{(-1)^n+\frac{1}{n}\right\}_{n=1}^{\infty}\quad\text{in }\mathbb{R}$$

does not converge but its set of limit points is $\{-1,1\}$.

Example 1.43. Consider the constant sequence $(1,1,\ldots)$ which converges to 1 but the set equals $\{1\}$, which does not have a limit point.

Proposition 1.10. Let (X,d) be a metric space. Then, the following are equivalent:

(i) a sequence $\{x_i\}_{i=1}^{\infty}$ converges to x in X

(ii) for all $\varepsilon > 0$, there exists $N > 0$ such that $d(x_i, x) < \varepsilon$ for all $i > N$

Proof. Suppose **(i)** holds. By Definition 1.21, for any open neighbourhood $U \ni x$, there exists $N > 0$ such that $x_k \in U$ for all $k > N$. Then, we can choose ε to be the radius of U and we are done. The proof of the reverse direction is the same. □

1.6. *Continuity*

Definition 1.22 (continuous map)**.** Let

$$X \text{ and } Y \quad \text{be topological spaces.}$$

A map $f : X \to Y$ is continuous if for any open set $U \subseteq Y$, $f^{-1}(U) \subseteq X$ is open.

Proposition 1.11. Let X and Y be topological spaces and $f : X \to Y$. Then, the following are equivalent:

(i) f is continuous

(ii) for all $A \subseteq X$, we have $f\left(\overline{A}\right) \subseteq \overline{f(A)}$

(iii) for any closed set $B \subseteq Y$, $f^{-1}(B) \subseteq X$ is closed

(iv) for any $x \in X$ and any open set $V \subseteq Y$ containing $f(x)$, there exists open $x \in U \subseteq X$ such that $f(U) \subseteq V$

Proof. We first prove **(i)** implies **(ii)**. Say we are given a continuous map $f : X \to Y$. Then, for any $A \subseteq X$, we have $\overline{f(A)} \subseteq Y$ is closed, so $Y \backslash \overline{f(A)}$ is open in Y. This implies $f^{-1}(Y \backslash \overline{f(A)} \subseteq X$ is open in X. As such, $f(A) \cap (Y \backslash \overline{f(A)}) = \emptyset$, which implies

$$A \cap f^{-1}(Y \backslash \overline{f(A)}) = \emptyset.$$

Hence,

$$\begin{aligned} \overline{A} \cap f^{-1}(Y \backslash \overline{f(A)}) &= \emptyset \\ f(\overline{A}) \cap (Y \backslash \overline{f(A)}) &= \emptyset \\ f(\overline{A}) &\subseteq \overline{f(A)} \end{aligned}$$

and the proof is complete.

We then prove **(ii)** implies **(iii)**. Let $B \subseteq Y$ be a closed set. Then, because $f^{-1}(B) \subseteq X$, replacing A with $f^{-1}(B)$ in **(ii)** yields

$$f(\overline{f^{-1}(B)}) \subseteq \overline{f(f^{-1}(B))} \subseteq \overline{B} = B.$$

This yields $\overline{f^{-1}(B)} \subseteq f^{-1}(B)$. Since the reverse inclusion follows by the definition of closure, we obtain equality, so it implies $f^{-1}(B) \subseteq X$.

We then prove **(iii)** implies **(iv)**. Let $x \in X$ and $V \subseteq Y$ be an open set containing $f(x)$. By **(iii)**, we have $f^{-1}(Y\backslash V) \subseteq X$ is closed. Together with $x \notin f^{-1}(Y\backslash V)$, we have $U = X\backslash(f^{-1}(Y\backslash V))$ is open and contains x. Moreover,

$$f(U) = f(X)\backslash(Y\backslash V) \subseteq Y \subseteq (Y\backslash V) = V.$$

Lastly, we prove **(iv)** implies **(i)**. Let $W \subseteq Y$ be open. By **(iv)**, for any $x \in f^{-1}(W)$, there exists an open set $U_x \subseteq X$ containing x such that $f(U_x) \subseteq W$. Then, notice that

$$f^{-1}(W) \subseteq \bigcup_{x\in f^{-1}(W)} U_x$$

and

$$f\left(\bigcup_{x\in f^{-1}(W)} U_x\right) = \bigcup_{x\in f^{-1}(W)} f(U_x) \subseteq W \quad \text{which implies} \quad \bigcup_{x\in f^{-1}(W)} U_x \subseteq f^{-1}(W).$$

These yield the statement

$$f^{-1}(W) = \bigcup_{x\in f^{-1}(W)} U_x \quad \text{being open in } X,$$

which completes the proof that **(iv)** implies **(i)**. □

Example 1.44 (MA4262 AY24/25 Sem 1 Homework 1). Show that with the discrete metric on a space X, the following properties hold:

(i) Every subset of X is open;
(ii) Every subset of X is closed;
(iii) Every subset of X has an empty boundary;
(iv) Every map $f : X \to X$ is continuous.

Solution. Recall that the discrete metric $d(x,y)$ is defined as follows:

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y; \\ 0 & \text{if } x = y \end{cases}$$

(i) Recall that a subset $A \subseteq X$ is open in (X,d) if and only if for all $y \in A$, there exists $\varepsilon > 0$ such that the open ball $\{x \in X : d(x,y) < \varepsilon\} \subseteq A$. Note that for every $\varepsilon \leq 1$, $B(x,\varepsilon) = \{x\}$ because $d(x,y) < 1$ if and only if $y = x$. Since $\{x\} \subseteq A$, it follows that A is open, so the first claim opens.

(ii) It suffices to show that for any $A \subseteq X$, its complement $X\backslash A$ is open. Since $X\backslash A \subseteq X$ and every subset of X is open, the result follows.

(iii) Let $A \subseteq X$. Recall that the boundary of A is the set difference of its closure $\overline{A}$ and its interior A°. Since $\overline{A}$ is the smallest closed set containing A, then by **(ii)**, $\overline{A} = A$. Similarly, A° is the largest open set contained in A, so by **(i)**, it is equal to A. So, the boundary of A is $A\backslash A = \emptyset$.

(iv) It suffices to show that for every open set $U \subseteq X$ (X here referring to the codomain), we have $f^{-1}(U) \subseteq X$ being open in X. This follows from **(i)** since every subset of X is open. □

Proposition 1.12 (pasting lemma). Let

$$X = A \cup B \quad \text{, where } A, B \subseteq X \text{ are both closed (or both open).}$$

Suppose $f : A \to Y$ and $g : B \to Y$ are continuous maps. If $f(x) = g(x)$ for all $x \in A \cap B$, then

$$h : X \to Y \quad \text{defined by} \quad h(x) = \begin{cases} f(x) & \text{if } x \in A; \\ g(x) & \text{if } x \in B \end{cases} \quad \text{is continuous.}$$

We will use this idea when studying homotopies and path homotopies in MA4266.

Proof. We will only prove for closed sets $G \subseteq Y$. Anyway, the proof works if we replace 'closed' with 'open'.

Suppose $G \subseteq Y$ is closed. Then, $f^{-1}(G) \subseteq A$ and $g^{-1}(B)$ are also closed. Since $X = A \cup B$, we have $f^{-1}(G), g^{-1}(G)$ being closed in X. As such, their union is also closed in X, i.e.

$$h^{-1}(G) = f^{-1}(G) \cup g^{-1}(G) \quad \text{is closed in } X.$$

The result follows. □

Definition 1.23 (pullback topology). Let $\mathcal{T}_Y$ be a topology on Y and let $f : X \to Y$ be a map. The pullback topology on X is defined to be

$$\mathcal{T}_X = \left\{ f^{-1}(U) : U \in \mathcal{T}_Y \right\}.$$

This is the coarsest topology on X such that f is continuous.

Example 1.45. Let $Y = \mathbb{R}$ be equipped with its standard topology and $X = \mathbb{Z}$. Define

$$f : \mathbb{Z} \to \mathbb{R} \quad \text{where} \quad f(n) = n.$$

For any open set $U \subseteq \mathbb{R}$, the pullback topology on $\mathbb{Z}$ would make every set $f^{-1}(U) \subseteq \mathbb{Z}$ open. However, since the preimage of any open set in $\mathbb{R}$ under f is always a discrete set, $\mathbb{Z}$ would inherit the discrete topology from $\mathbb{R}$ in this case.

Definition 1.24 (uniform continuity). Let

$$(X, d_X) \text{ and } (Y, d_Y) \quad \text{be two metric spaces.}$$

A map $f : X \to Y$ is uniformly continuous on X if for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \varepsilon$.

Example 1.46. Let (X, d) be a metric space and $A \subseteq X$ be non-empty. Then, the function $f : X \to \mathbb{R}$ defined by $f(x) = d(x, A)$ is uniformly continuous.

In fact, note that for any $x, y \in X$, we have

$$d(x,A) = \inf_{a\in A} d(x,a) \leq \inf_{a\in A} (d(x,y) + d(y,a)) = d(x,y) + d(y,A).$$

This implies that

$$|d(x,A) - d(y,A)| \leq d(x,y).$$

By choosing $\delta = \varepsilon$ in Definition 1.24, we complete the checking process.

Proposition 1.13. Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is uniformly continuous if and only if for any two sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ in X such that

$$\lim_{i\to\infty} d_X(x_i, y_i) = 0, \quad \text{we have} \quad d_Y(f(x_i), f(y_i)) = 0.$$

The proof of this result is trivial. One can apply the definition directly to prove the forward direction; the reverse direction can be proven using contradiction, i.e. suppose on the contrary that there exists $\varepsilon > 0$ such that for all $i \in \mathbb{N}$, there exists $x_i, y_i \in X$ such that

$$d_X(x_i, y_i) < \frac{1}{i} \quad \text{but} \quad d_Y(f(x_i), f(y_i)) > \varepsilon,$$

which yields a contradiction.

One should recall the following from MA3210:

Definition 1.25 (pointwise convergence and uniform convergence). Let $f_i : X \to Y$ be a sequence of maps from a set X to a metric space (Y, d).

- **(i)** f_i converges pointwise to $f : X \to Y$ if $f_i(x) \to f(x)$ for all $x \in X$
- **(ii)** f_i converges uniformly to $f : X \to Y$ if for any $\varepsilon > 0$, there exists $N > 0$ such that for all $i \geq N$ and $x \in X$, we have $d(f_i(x), f(x)) < \varepsilon$

1.7. *Product of Topological Spaces*

Definition 1.26 (product and projection map). Let $\{X_\alpha\}_{\alpha\in\Lambda}$ be non-empty sets. Consider the product

$$\prod_{\alpha\in\Lambda} X_\alpha = \{(x_\alpha)_{\alpha\in\Lambda} : x_\alpha \in X_\alpha \text{ for all } \alpha \in \Lambda\}$$

For all $\alpha \in \Lambda$, the map

$$\pi_{X_\alpha} : \prod_{\alpha\in\Lambda} X_\alpha \to X_\alpha \text{ defined by } (x_\alpha)_{\alpha\in\Lambda} \mapsto x_\alpha \quad \text{is the projection to the } \alpha^{\text{th}} \text{ factor.}$$

Note that if $\alpha_0 \in \Lambda$, then for any $U \subseteq X_{\alpha_0}$, we have

$$\pi_{X_{\alpha_0}}^{-1}(U) = \left\{ (x_\alpha)_{\alpha\in\Lambda} \in \prod_{\alpha\in\Lambda} X_\alpha : x_{\alpha_0} \in U \right\}.$$

This is simply known as the inverse image of the projection map.

Moreover, if $X_\alpha = X$ for all $\alpha \in \Lambda$, then the correspondence

$$\prod_{\alpha\in\Lambda} X_\alpha \quad \text{and} \quad \text{the set of maps } f : \Lambda \to X \text{ defined by } \alpha \mapsto x_\alpha$$

is defined by

$$(x_\alpha)_{\alpha\in\Lambda} \mapsto (f : \alpha \mapsto x_\alpha).$$

Definition 1.27 (product topology). Suppose $(X_\alpha, \mathcal{T}_\alpha)_{\alpha\in\Lambda}$ are topological spaces. The product topology on $\prod_{\alpha\in\Lambda} X_\alpha$ is the topology generated by the subbasis

$$\mathcal{S} = \left\{ \pi_{X_\alpha}^{-1}(U_\alpha) : \alpha \in \Lambda, U_\alpha \in \mathcal{T}_\alpha \right\}.$$

Definition 1.28 (box topology). Suppose $(X_\alpha, \mathcal{T}_\alpha)_{\alpha\in\Lambda}$ are topological spaces. The box topology on $(X_\alpha, \mathcal{T}_\alpha)_{\alpha\in\Lambda}$ is the topology generated by

$$\mathcal{B} = \left\{ \prod_{\alpha\in\Lambda} U_\alpha \subseteq X_\alpha \text{ is open} \right\}.$$

Remark 1.5. The product topology and box topology are the same for finite product but different for infinite product.

Proposition 1.14. Let $\{X_\alpha\}_{\alpha\in\Lambda}$ be topological spaces. For any $\alpha \in \Lambda$, let

$$\pi_{X_\alpha} : \prod_{\alpha\in\Lambda} X_\alpha \to X_\alpha \quad \text{be} \quad \text{the projection onto the } \alpha^{\text{th}} \text{ factor.}$$

Then, the following hold:

(i) the product topology on $\prod_{\alpha\in\Lambda} X_\alpha$ is the coarsest topology such that π_{X_α} is continuous for any $\alpha \in \Lambda$

(ii) for any topological space Y and $\alpha \in \Lambda$, let $f_\alpha : Y \to X_\alpha$. The map

$$f = \prod_{\alpha\in\Lambda} f_\alpha : Y \to \prod_{\alpha\in\Lambda} X_\alpha \quad \text{defined by} \quad y \mapsto (f_\alpha(y))_{\alpha\in\Lambda}$$

is continuous if and only if f_α is continuous for every $\alpha \in \Lambda$.

Proof. We first prove **(i)**. Suppose $\alpha \in \Lambda$ and $U \subseteq X_\alpha$ be open. Then, $\pi_{X_\alpha}^{-1}(U)$ lies in the subbasis that generates the product topology and hence is open. So, for every $\alpha \in \Lambda$, π_{X_α} is a continuous map to the α^{th} factor.

Note that a topology on

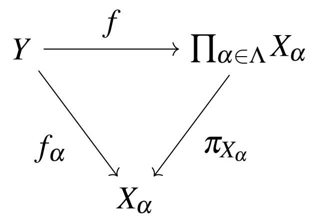
$$\prod_{\alpha\in\Lambda} X_\alpha \quad \text{contains} \quad \mathcal{S} = \left\{\pi_{X_\alpha}^{-1}(U_\alpha) : \alpha \in \Lambda, U_\alpha \in \mathcal{T}_\alpha\right\}$$

is equivalent to it containing the basis

$$\mathcal{B} = \left\{\bigcap_{\alpha\in\Lambda'} \pi_{X_\alpha}^{-1}(U_\alpha) : \Lambda' \subseteq \Lambda \text{ is finite}, U_\alpha \subseteq X_\alpha \text{ is open}\right\}.$$

By Remark 1.1, the product topology is the coarsest one that contains $\mathcal{B}$, so it is the coarsest one that contains $\mathcal{S}$, and so it is the coarsest one for which π_{X_α} is continuous for every $\alpha \in \Lambda$.

We then prove **(ii)**, starting with the forward direction. Note that $f_\alpha = \pi_{X_\alpha} \circ f : Y \to X_\alpha$ as shown in the following commutative diagram:



By **(i)**, π_{X_α} is continuous. Since f is also continuous, we conclude that f_α is continuous for every $\alpha \in \Lambda$ (recall that the composition of composite maps is also continuous from MA1100/MA2108).

As for the reverse direction, pick any finite $\Lambda' \subseteq \Lambda$. Then,

$$f^{-1}\left(\bigcap_{\alpha\in\Lambda'} \pi_{X_\alpha}^{-1}(U_\alpha)\right) = \bigcap_{\alpha\in\Lambda'} (\pi_{X_\alpha} \circ f)^{-1}(U_\alpha) = \bigcap_{\alpha\in\Lambda'} f_\alpha^{-1}(U_\alpha)$$

which is open since f_α is continuous for every $\alpha \in \Lambda$. This in particular shows that

$$f^{-1}(B) \text{ is open} \quad \text{for every} \quad B \text{ in the basis } \mathcal{B}.$$

We conclude that f is continuous. $\square$

Example 1.47 (component functions of f are continuous but f not continuous)**.** Let $\mathbb{R}^{\mathbb{N}}$ be the countable Cartesian product of $\mathbb{R}$ with itself, i.e. the set of all sequences in $\mathbb{R}$. Equip $\mathbb{R}$ with the standard topology and $\mathbb{R}^{\mathbb{N}}$ with the box topology. For the latter, say we have a basis element for the box topology. Then, it comprises open sets of the form

$$U_1 \times U_2 \times U_3 \times \ldots \quad \text{where } U_i \subseteq \mathbb{R} \text{ is open in } \mathbb{R} \text{ for all } i.$$

Define

$$f : \mathbb{R} \to \mathbb{R}^{\mathbb{N}} \quad \text{via} \quad x \mapsto (x, x, x, \ldots).$$

All the component functions are the identity which is continuous. However, f is not a continuous map. To see why, let

$$U = \prod_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n} \right).$$

Suppose on the contrary that f is continuous. Since $f(0) = (0,0,0,\ldots) \in U$, then there exists $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subseteq f^{-1}(U)$. However, this implies that

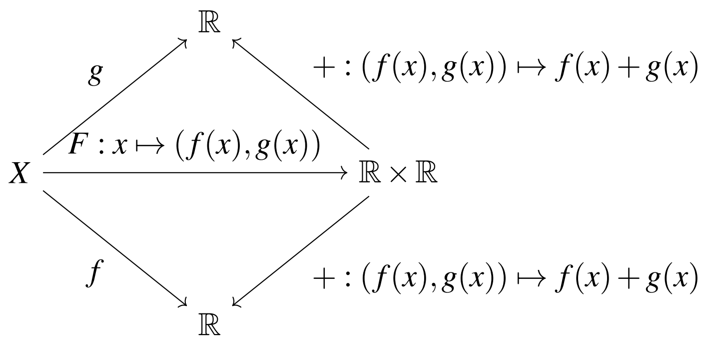
$$f\left(\frac{\varepsilon}{2}\right) = \left(\frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \ldots\right) \in U.$$

This is false since $\varepsilon/2 > 1/n$ for n sufficiently large, i.e. choose n such that $n = \lceil 2/\varepsilon \rceil$. So, f is not continuous even though all its component functions are.

In Example 1.47, having said that, if we replace the example with $f : \mathbb{R} \to \mathbb{R}^2$ via $f(x) = (x,x)$, then f is a continuous map since the intersection of two open sets in $\mathbb{R}$ is also an open set in $\mathbb{R}$. However, this property fails when the codomain is $\mathbb{R}^{\mathbb{N}}$.

Corollary 1.1 (operations on continuous maps). Let X be a topological space and $f, g : X \to \mathbb{R}$ be continuous. Then, $f+g$, $f-g$ and $f \cdot g$ are continuous. Also, if $0 \notin g$, then f/g is continuous.

Proof. This seems like a trivial result from MA2108 but now the domain is generalised to arbitrary topological spaces. Firstly, define the map $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as $(a,b) \mapsto a+b$, which is continuous with respect to the standard topology on $\mathbb{R}$. Let $F : X \to \mathbb{R} \times \mathbb{R}$ be defined by $x \mapsto (f(x), g(x))$. Then, F is continuous by **(ii)** of Proposition 1.14. In particular, the following diagram commutes:



As such, $f+g = + \circ F$, so $f+g$ is continuous. Using this method, the continuity properties involving the other operations can be deduced similarly. □

Example 1.48. Let $n \in \mathbb{Z}^+$ and decompose $n = m_1 + \ldots + m_k$, where $m_i \in \mathbb{Z}^+$ for every $1 \le i \le k$. Prove that the product topology of standard topologies on $\mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_k} = \mathbb{R}^n$ is the standard topology on $\mathbb{R}^n$.

Solution. We first prove that the product topology is finer than the standard topology. Note that a basis $\mathcal{B}$ for the standard topology on $\mathbb{R}^n$ is of the form $U_{m_1} \times \ldots \times U_{m_k}$, where each U_{m_i} is an open ball in

$\mathbb{R}^{m_i}$. As such, each U_{m_i} is of the form (a_{m_i}, b_{m_i}). Hence, $\mathcal{B}$ has elements of the form

$$(a_{m_1}, b_{m_1}) \times \ldots \times (a_{m_k}, b_{m_k})$$

As mentioned, each (a_{m_i}, b_{m_i}) is open in $\mathbb{R}^{m_i}$, so the product of these sets is open in the product topology.

Then, we prove that the standard topology is finer than the product topology. Let $B = U_{m_1} \times \ldots \times U_{m_k}$ be an element of the basis $\mathcal{B}$ of the product topology, where each U_{m_i} is open in $\mathbb{R}^{m_i}$. As each U_{m_i} is an open ball, then B can be written as the union of sets of the form $(a_{m_1}, b_{m_1}) \times \ldots \times (a_{m_k}, b_{m_k})$, each being an open set in $\mathbb{R}^{m_i}$ respectively. The result follows. □

Example 1.49 (MA3209 AY24/25 Sem 1 Homework 1). Let $H = [0,1]^{\mathbb{N}}$ be the Hilbert cube, with the product topology. Thus the sets of the form

$$B = \left\{ \{x_n\}_{n \in \mathbb{N}} \in H : \exists N \in \mathbb{N} \text{ and open intervals } I_j \subseteq [0,1], j \leq N \text{ such that } x_j \in I_j \right\}$$

constitute a basis of this topology.

(a) Show that

$$d(x,y) = \sum_{n \in \mathbb{N}} 2^{-n} |x_n - y_n| \quad \text{is a metric on } H.$$

(b) Show that d induces the product topology.

Solution.

(a) We will only prove that d satisfies the triangle inequality. Suppose

$$d(x,y) = \sum_{n \in \mathbb{N}} 2^{-n} |x_n - y_n| \quad \text{and} \quad d(y,z) = \sum_{n \in \mathbb{N}} 2^{-n} |y_n - z_n|.$$

Then,

$$\begin{aligned} d(x,z) &= \sum_{n \in \mathbb{N}} 2^{-n} |x_n - z_n| \\ &= \sum_{n \in \mathbb{N}} 2^{-n} |(x_n - y_n) + (y_n - z_n)| \\ &\leq \sum_{n \in \mathbb{N}} 2^{-n} |x_n - y_n| + \sum_{n \in \mathbb{N}} 2^{-n} |y_n - z_n| \quad \text{by triangle inequality} \\ &= d(x,y) + d(y,z) \end{aligned}$$

(b) We first prove that the topology induced by the metric is finer than the product topology. Consider some element B in the product topology. Then, B takes the following form:

$$I_1 \times \ldots \times I_N \times [0,1]^{\mathbb{N} \setminus \{1,\ldots,N\}}$$

Here, each $I_j \subseteq [0,1]$ is an open interval. For any $\mathbf{x} \in B$, we must have $x_j \in B$ for all $1 \leq j \leq N$. Since each I_j is an open interval, then there exists $\varepsilon_j > 0$ such that $(x - \varepsilon_j, x + \varepsilon_j) \subseteq I_j$. Define

$$r = \min \left\{ \frac{\varepsilon_j}{2^j} : 1 \leq j \leq N \right\}.$$

Then, we claim that $\mathbf{x} \in B_r(\mathbf{x}) \subseteq B$. The fact that$\mathbf{x}$ is an element of the open ball $B_r(\mathbf{x})$ is obvious. We shall prove that the subset inclusion holds. For any $\mathbf{y} \in B_r(\mathbf{x})$, we have $d(\mathbf{x},\mathbf{y}) < r$ so

$$\frac{|x_n - y_n|}{2^n} \leq d(\mathbf{x},\mathbf{y}) < r \leq \frac{r_n}{2^n} \quad \text{which implies} \quad |x_n - y_n| < r_n.$$

Hence, $y_n \in (-r_n + x_n, r_n + x_n)$, which asserts that the topology induced by the metric is finer than the product topology.

We then prove that the product topology is finer than the topology induced by the metric. Note that the topology on H induced by d is the topology generated by $\mathcal{B}_d$, for which this basis denotes the set of open balls $B_r(\mathbf{x})$, where $x \in X$ and $r > 0$. Let $B_\varepsilon(\mathbf{x}) \in \mathcal{B}_d$, where $\mathbf{x} \in H$. Suppose $\mathbf{y} \in B_\varepsilon(\mathbf{x})$. Since $B_\varepsilon(\mathbf{x})$ is an open ball, then there exists $r > 0$ such that $B_r(\mathbf{y}) \subseteq B_\varepsilon(\mathbf{x})$.

Choose $N \in \mathbb{N}$ such that $1/2^N < r/2$. Since every basis element B in the product topology is of the form

$$I_1 \times \ldots \times I_N \times \prod_{k>N} I_k \quad \text{for } 1 \leq k \leq N,$$

then let

$$I_k = B_{r/2N}(y_k) \quad \text{and} \quad I_k = [0,1] \text{ for } k > N.$$

Now, let $w \in B$, be any arbitrary element in the basis. Then, one can prove that $d(w,y) < r$. So, we have $\mathbf{y} \in B \subseteq B_r(\mathbf{y}) \subseteq B_\varepsilon(\mathbf{x})$, implying that the product topology is finer than the topology induced by the metric. □

1.8. *Product of Metric Spaces*

Definition 1.29 (Manhattan metric and Chebyshev metric). Suppose $(X_1, d_{X_1}), \ldots, (X_n, d_{X_n})$ are metric spaces. The following are two common metrics on $X_1 \times \ldots \times X_n$:

- **(i) Manhattan metric:** $d_1((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = d_{X_1}(x_1,y_1) + \ldots + d_{X_n}(x_n,y_n)$
- **(ii) Chebyshev metric:** $d_\infty((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \max\{d_{X_i}(x_i,y_i) : 1 \leq i \leq n\}$

Note that the Manhattan metric d_1 is also known as the product metric or taxicab metric and it is defined as the sum of the individual distances across each coordinate; the Chebyshev metric d_∞ is also known as the maximum metric or the supremum metric and it takes the maximum of the individual distances between corresponding points across the coordinates.

Moreover, in Definition 1.29, if $X_i = \mathbb{R}$ and d_i is the Euclidean metric for all $1 \leq i \leq n$, then d_1 and d_∞ are known as the L^1-metric and L^∞-metric respectively.

As for the infinite product case, there is no natural metric on an uncountable product of metric spaces.

Thus, we stick to countable products. As an example, if we let $(X_i, d_{X_i})_{i=1}^{\infty}$ be metric spaces, given the metric d_∞ as mentioned, we can define

$$d_\infty : \prod_{i=1}^{\infty} X_i \times \prod_{i=1}^{\infty} X_i \to \mathbb{R} \quad \text{by} \quad d_\infty(x,y) = \sup\left\{d_{X_i}(x_i,y_i) : i \in \mathbb{Z}^+\right\}.$$

However, this is not well-defined as $d_{X_i}(x_i,y_i)$ might be unbounded as $i \to \infty$.

Example 1.50 (infinite supremum for infinite product of metric spaces)**.** Let (X_i, d_{X_i}) be metric spaces. For each $i \in \mathbb{Z}^+$, let $X_i = \mathbb{R}$ be the real number line equipped with the standard Euclidean metric $d_{X_i}(x_i,y_i) = |x_i - y_i|$. Consider the sequences $x = \{x_i\}_{i=1}^{\infty}$ and $y = \{y_i\}_{i=1}^{\infty}$ in the product space $\mathbb{R} \times \mathbb{R} \times \ldots$, which is an infinite product. Here, $x_i = i$ and $y_i = 0$ for all $i \in \mathbb{Z}^+$.

For every $i \in \mathbb{Z}^+$, the distance between the components of x and y is $d_{X_i}(x_i,y_i) = |x_i - y_i| = i$. However, $d_\infty(x,y)$ is the supremum of $d_{X_i}(x_i,y_i)$ over all $i \in \mathbb{Z}^+$, which is infinity. However, this means that d_∞ is not well-defined.

Definition 1.30 (open ball)**.** Let (X,d) be a metric space. For any $x \in X$, define

$$B_r^d(x) \quad \text{to be} \quad \text{the open ball of radius } r \text{ centred at } x \text{ in the metric space } (X,d).$$

So,

$$B_r^d(x) = \{y \in X : d(x,y) < r\}$$

includes all points $y \in X$ such that the distance between x and y (as measured by the metric d) is strictly less than r.

Proposition 1.15 (bounded metric)**.** Let (X,d) be a metric space. Then

$$\rho : X \times X \to \mathbb{R} \quad \text{given by} \quad \rho(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

is a metric with diameter less than 1. Furthermore, ρ and d induce the same topology on X.

Proof. First, we verify that ρ is a metric. Note that the non-negativity, positive definiteness and symmetry properties are obvious so we will only prove the triangle inequality. Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be given by $f(t) = t/(1+t)$. In fact, the codomain $\mathbb{R}$ can be restricted to $\mathbb{R}_{\geq 0}$ but this is permitted once we verified that ρ is a metric.

One can easily show that f is increasing, concave down and $f(0) = 0$, so for any $a, b \geq 0$, we have

$f(a+b) \leq f(a)+f(b)$. Hence,

$$\begin{aligned}\rho(x,z) &= f(d(x,z)) \quad \text{by definition of } \rho \\ &\leq f(d(x,y)+d(y,z)) \quad \text{since } d \text{ satisfies the triangle inequality} \\ &= f(d(x,y))+f(d(y,z)) \quad \text{since } f(a+b) \leq f(a)+f(b) \\ &= \rho(x,y)+\rho(y,z)\end{aligned}$$

so ρ satisfies the triangle inequality. Moreover, $\rho(x,y) < 1$ for all $x,y \in X$. So, we conclude that the diameter of ρ is less than 1. In other words, the largest distance between any two $x,y \in X$ is less than 1.

In order to prove that d and ρ induce the same topology on X, we will make use of the following result: if $x,y \in X$ such that $d(x,y) < 1$, then

$$\rho(x,y) < d(x,y) < 2\rho(x,y).$$

In particular, for every $x \in X$ and $0 < r < 1$, we have

$$B^{\rho}_{r/2}(x) \subset B^{d}_{r}(x) \subset B^{\rho}_{r}(x)$$

The proof of this result is obvious. Now, let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies induced by d and ρ respectively. Just as before, it suffices to show that $\mathcal{T}$ is finer than $\mathcal{T}'$ and $\mathcal{T}'$ is finer than $\mathcal{T}$. Note that for any $x \in X$, if $x \in B^{d}_{r}(y)$, then there exists $\delta \in (0,1)$ such that

$$x \in B^{d}_{\delta}(x) \subseteq B^{d}_{r}(y) \quad \text{so} \quad x \in B^{\rho}_{\delta/2}(x) \subseteq B^{d}_{r}(y).$$

The second statement follows by the claim that was established earlier. In particular, we replace r with δ. This shows that $\mathcal{T}$ is finer than $\mathcal{T}'$. By a symmetric argument, one can show that $\mathcal{T}'$ is finer than $\mathcal{T}$. □

Proposition 1.16. Let $(X_i, d_{X_i})_{i=1}^{\infty}$ be metric spaces for all i and let

$$\rho_{X_i}(x,y) = \frac{d_{X_i}(x,y)}{1+d_{X_i}(x,y)} \quad \text{for all } x,y \in X_i.$$

Then,

$$d : \prod_{i=1}^{\infty} X_i \times \prod_{i=1}^{\infty} X_i \to \mathbb{R} \quad \text{given by} \quad d(x,y) = \sup\left\{\frac{1}{i} \cdot \rho_{X_i}(x_i,y_i) : i \in \mathbb{Z}\right\}$$

is a metric that induces the product topology on $\prod_{i=1}^{\infty} X_i$.

Proof. We will only prove that d is a metric. Again, the non-negativity, positive definiteness, and symmetry are obvious, so we will only prove that d satisfies the triangle inequality. We have

$$\begin{aligned} d(x,y)+d(y,z) &= \sup\left\{\frac{1}{i}\rho_{X_i}(x_i,y_i) : i\in\mathbb{Z}^+\right\} + \sup\left\{\frac{1}{i}\rho_{X_i}(y_i,z_i) : i\in\mathbb{Z}^+\right\} \\ &=\geq \sup\left\{\frac{1}{i}\cdot\left(\rho_{X_i}(x_i,y_i)+\rho_{X_i}(y_i,z_i)\right) : i\in\mathbb{Z}^+\right\} \quad \text{since } \sup(A+B)=\sup A+\sup B \\ &\geq \sup\left\{\frac{1}{i}\cdot\rho_{X_i}(x_i,z_i) : i\in\mathbb{Z}^+\right\} = d(x,z) \end{aligned}$$

which shows that d is a metric. □

1.9. *Quotient of Topological Spaces*

Definition 1.31 (quotient map)**.** Let X and Y be topological spaces. A surjective map $p : X \to Y$ is a quotient map if

$$V \subseteq Y \text{ is open} \quad \text{if and only if} \quad p^{-1}(V) \subseteq X \text{ is open.}$$

[open and closed maps] Let X and Y be topological spaces. A continuous map

$$f : X \to Y \text{ is open} \quad \text{if} \quad f(U) \text{ is open for any open } U \subseteq X.$$

We obtain the definition of a closed map by replacing all 'open' with 'closed'.

Proposition 1.17. If

$$f : X \to Y \text{ and } g : Y \to Z \quad \text{are both quotient maps,}$$

then $g\circ f : X \to Z$ is also a quotient map. Note that 'quotient' can be replaced with 'open' or 'closed' and the proposition would still hold.

Example 1.51 (map that is closed but not open)**.** Let $p : [0,1]\cup[2,3] \to [0,2]$ be a map defined by

$$p(x) = \begin{cases} x & \text{if } x\in[0,1]; \\ x-1 & \text{if } x\in[2,3]. \end{cases}$$

By the pasting lemma (Proposition 1.12), p is a continuous map. To see why, $[0,1]\cup[2,3]$ is the union of two closed intervals $[0,1]$ and $[2,3]$. On the interval $[0,1]$, the map $p(x) = x$ is continuous; on the interval $[2,3]$, the map $p(x) = x-1$ is also continuous. As the intersection of these two intervals is $\emptyset$, there is no conflicting definition at any point.

Moreover, p is a closed map. To see why, let $C \subseteq [0,1]\cup[2,3]$ and we have to show that $p(C)$ is also closed in $[0,2]$. There are three cases to consider — firstly, if $C\subseteq[0,1]$, secondly, if $C\subseteq[2,3]$ and thirdly, if C is a closed set that intersects $[0,1]$ and $[2,3]$. Arguing that p is a closed map for the first two cases is trivial. As for the third case, we have

$$p(C) = p(C \cap [0,1]) \cup p(C \cap [2,3]).$$

Note that $C \cap [0,1]$ and $C \cap [2,3]$ are closed since the intersection of two closed sets is also closed. So, $p(C \cap [0,1])$ and $p(C \cap [2,3])$ are closed sets. Hence, $p(C)$ is closed in $[0,2]$.

As p is continuous, surjective and closed, we conclude that p is a quotient map.

Having said that, p is not an open map. To see why, let $B = (0,1]$, so B is open with respect to the subspace topology in $[0,1] \cup [2,3]$. However, $p((0,1]) = (0,1] \subseteq [0,2]$ which is not open. To see why, in the standard topology on $[0,2]$, open sets are typically intervals that do not include their endpoints.

Example 1.52 (map that is open but not closed). A classic example of a map that is open but not closed is the projection map

$$\pi : \mathbb{R}^2 \to \mathbb{R} \quad \text{where} \quad \pi(x,y) = x.$$

Here, $\mathbb{R}^2$ is equipped with the standard topology. To see why the claim is true, we first show that π is an open map. Note that π sends open sets in $\mathbb{R}^2$ to open sets in $\mathbb{R}$. For example, consider an open set $U = (a,b) \times (c,d) \subseteq \mathbb{R}^2$. Under π, its image is the open interval (a,b), which is open in $\mathbb{R}$. Think of this as projecting the base of a rectangle in $\mathbb{R}^2$ onto a segment of the x-axis.

However, π is not a closed map. To see why, consider the closed set

$$C = \left\{ \left(\frac{1}{x}, x \right) : x > 0 \right\} \subseteq \mathbb{R}^2.$$

C contains no limit points so by vacuous truth, $F \subseteq \mathbb{R}^2$ is a closed set. The image of C under π is the set

$$\left\{ x : \frac{1}{x} > 0 \right\} = (0, \infty)$$

which is not closed in $\mathbb{R}$ as it does not contain its limit point 0.

Example 1.53 (surjective and continuous map that is neither open nor closed). Let

$$X = \mathbb{R}^2 \backslash \{(x,y) : 0 \leq x < 1, 0 < y < 1\} \quad \text{and} \quad f : X \to \mathbb{R} \text{ be defined as } f(x,y) = x.$$

Note that f is surjective and continuous. It is clear that f is continuous that it is the restriction of the projection of $\mathbb{R}^2$ onto its first factor which is continuous.

However, f is not open. To see why,

$$f\left(B_{1/3}\left(\left(1, \frac{1}{2} \right) \right) \cap X \right) = \left[1, \frac{4}{3} \right) \subseteq \mathbb{R} \quad \text{is not open.}$$

Also, f is not closed. To see why,

$$f\left(\overline{B_{1/3}\left(\left(0,\frac{1}{2}\right)\right)}\cap X\right)=\left[-\frac{1}{3},0\right)\subseteq\mathbb{R}\quad\text{is not closed.}$$

Having said that, f is a quotient map. To see why, let $A\subseteq\mathbb{R}$ and suppose $f^{-1}(A)\subseteq X$ is open. Then,

$$f^{-1}(A)\cap(\mathbb{R}\times(5,6))\subseteq\mathbb{R}\times(5,6)\quad\text{is open}$$

since this is the intersection of open sets. Moreover,

$$f^{-1}(A)\cap(\mathbb{R}\times(5,6))=A\times(5,6).$$

Moreover, the set

$$\{U\times V: U\subseteq\mathbb{R}\text{ is open}, V\subseteq(5,6)\text{ is open}\}\quad\text{is a basis for }\mathbb{R}\times(5,6).$$

Let $x\in A$, then $A\times(5,6)\subseteq\mathbb{R}\times(5,6)$ implies that there exists $U\subseteq\mathbb{R}$ and $V\subseteq(5,6)$ both open such that $(x,11/2)\in U\times V\subseteq A\times(5,6)$, which shows that $x\in U\subseteq A$, so A is open.

Let X and Y be topological spaces. It is of interest to discuss the following question:

how different are quotient maps from open maps or closed maps?

A continuous surjection $q:X\to Y$ is a quotient map if a subset $U\subseteq Y$ is open in Y if and only if its preimage $q^{-1}(U)$ is open in X. This property ensures that Y is endowed with the quotient topology induced by q. On the other hand, open maps (maps f which send every open set in X to an open set in Y) need not be surjections or continuous, although they are generally continuous maps. The definition of a closed map is similar to that of an open map — they are not required to be surjections or continuous.

Definition 1.32 (saturated set). Let

$$f:X\to Y\text{ be a surjective continuous map}\quad\text{and}\quad A\subseteq X.$$

The set A is saturated with respect to f if

$$A=f^{-1}(S)\quad\text{for some }S\subseteq Y.$$

Note that Definition 1.32 is equivalent to saying that $A=f^{-1}(f(A))$.

Proposition 1.18. Let $f:X\to Y$ be a surjective continuous map. Then, the following hold:

(i) f is a quotient map if and only if f sends every saturated open set to an open set. The same property holds if we replace 'open' with 'closed'.

(ii) If f is a quotient map and $A \subseteq X$ is saturated and open, then the restriction map

$$f|_A : A \to f(A) \quad \text{is also a quotient map.}$$

Similar to **(i)**, if we replace 'open' with 'closed', the property still holds.

Proof. We first prove **(i)**. We prove the forward direction. Suppose $A \subseteq X$ is open and saturated. Earlier, we mentioned that an equivalent way of stating Definition 1.32 is that $A = f^{-1}(f(A))$, so $f(A) \subseteq Y$ is open as a quotient map. Since

$$A \text{ is saturated and open} \quad \text{and} \quad f(A) \text{ is open},$$

the result follows.

We then prove the reverse direction of **(i)**. Let $U \subseteq Y$ and suppose $f^{-1}(U)$ is open. Since $f^{-1}(U)$ is saturated, then $f(f^{-1}(U)) \subseteq X$ is open. Since f is surjective and $U = f(f^{-1}(U))$, it follows that f is a quotient map.

Now, we prove **(ii)**. By the preamble,

$$f|_A : A \to f(A) \quad \text{is surjective and continuous.}$$

Let $B \subseteq A$ be open and saturated with respect to $f|_A$. By **(i)**, it suffices to prove that $f|_A(B) \subseteq f(A)$ is open. Note that

$$B = (f|_A)^{-1}(f|_A(B)) = f^{-1}(f(B)) \quad \text{as } A \text{ is saturated with respect to } f.$$

Since $B \subseteq A$ and $A \subseteq X$ is open, then $B \subseteq X$ is open. By **(i)**, we know that $f(B)$ is open and thus, $f|_A(B) = f(B)$ is also open in Y. The result follows. □

Proposition 1.19. If X is a topological space, $A \subseteq X$ and $p : X \to A$ is surjective. Then, there exists a unique topology on A (called the quotient topology) such that p is a quotient map.

Proof. Let

$$\mathcal{T} = \left\{ U \subseteq A : p^{-1}(U) \subseteq X \text{ is open} \right\}.$$

Then, $\emptyset, A \in \mathcal{T}$. Suppose $\{U_\alpha\}_{\alpha \in \Lambda} \subseteq \mathcal{T}$. Then,

$$p^{-1}\left(\bigcup_{\alpha \in \Lambda} U_\alpha \right) = \bigcup_{\alpha \in \Lambda} p^{-1}(U_\alpha) \subseteq X \quad \text{is open}$$

which shows that the arbitrary union is contained in $\mathcal{T}$.

Next, suppose $U_1, \ldots, U_n \in \mathcal{T}$. Then,

$$p^{-1}\left(\bigcap_{\alpha\in\Lambda} U_\alpha\right) = \bigcap_{\alpha\in\Lambda} p^{-1}(U_\alpha) \subseteq X \quad \text{is open}$$

which shows that the finite intersection is contained in $\mathcal{T}$.

These show that $\mathcal{T}$ is a topology. We then show that the topology is unique. By definition, $p : X \to (A, \mathcal{T})$ is a quotient map. Suppose $\mathcal{T}'$ is another topology on A such that $p : X \to (A, \mathcal{T}')$ is a quotient map. So,

$$U \in \mathcal{T}' \quad \text{if and only if} \quad p^{-1}(U) \subseteq X \text{ is open} \quad \text{if and only if} \quad U \in \mathcal{T}.$$

This shows that $\mathcal{T} = \mathcal{T}'$. □

Example 1.54. Let $p : \mathbb{R} \to \{a, b, c\}$ be a map defined as

$$x \mapsto \begin{cases} a & \text{if } x > 0; \\ b & \text{if } x = 0; \\ c & \text{if } x < 0. \end{cases}$$

Then, the quotient topology on $\{a, b, c\}$ is

$$\mathcal{T} = \{\{a\}, \{c\}, \{a, c\}, \{a, b, c\}, \emptyset\}.$$

Definition 1.33 (quotient space)**.** Let X be a topological space and let X^* be the cells of a partition of X. Let $p : X \to X^*$ be the surjective map that sends each point in X to the subset that contains it. X' equipped with the quotient topology induced by p is a quotient space of X.

Example 1.55 (partitioning $\mathbb{R}$)**.** Partition

$$X = \mathbb{R} = \mathbb{R}^- \cup \{0\} \cup \mathbb{R}^+.$$

Then, $X^* = \{\mathbb{R}^-, \{0\}, \mathbb{R}^+\}$. To see why, recall that X^* represents the set of equivalence classes of X under an equivalence relation induced by the given partition. When we partition a set X into disjoint subsets, we are effectively defining an equivalence relation where elements are considered equivalent if they belong to the same subset of the partition.

The partition induces an equivalence relation $\sim$ on X where two elements $x, y \in X$ are equivalent if they belong to the same subset of the partition. The equivalence classes under this relation are precisely the subsets in our partition, i.e.

$$[x] = \begin{cases} \mathbb{R}^- & \text{if } x \text{ is negative;} \\ \{0\} & \text{if } x = 0; \\ \mathbb{R}^+ & \text{if } x \text{ is positive.} \end{cases}$$

Example 1.56 (partitioning $\mathbb{D}$). Let $X = \mathbb{D}$ be the closed unit disc, i.e.

$$X = \left\{(x,y) : x^2 + y^2 \leq 1\right\}.$$

We can decompose it as the following union:

$$\left\{(x,y) : x^2 + y^2 = 1\right\} \cup \bigcup_{(x,y):x^2+y^2<1} \{(x,y)\}$$

Then,

$$X^* = \left\{\{(x,y)\} : x^2 + y^2 < 1\right\} \cup \left\{\left\{(x,y) : x^2 + y^2 = 1\right\}\right\}.$$

Example 1.57. Let $X = \mathbb{R}$ and define a surjective map

$$p : X \to X^* \quad \text{where} \quad p : x \mapsto x + n \text{ for some } n \in \mathbb{Z} \text{ such that } x + n \in [0,1).$$

Then, such n is unique for a fixed $x \in \mathbb{R}$. In this setting, $X^* = [0,1)$. In fact, the process of adding an integer n to x to bring it into $[0,1)$ is equivalent to considering real numbers modulo 1. We define an equivalence relation $\sim$ on $\mathbb{R}$ where

$$x \sim y \quad \text{if and only if} \quad x - y \in \mathbb{Z}.$$

Hence, the equivalence class of x is $[x] = \{y \in \mathbb{R} : y \sim x\} = x + \mathbb{Z}$. We may also identify X^* as the unit circle $\mathbb{S}^1$ or $\mathbb{R}/\mathbb{Z}$.

2. Topological and Metric Properties of Spaces

2.1. T_1 and T_2 Spaces

Definition 2.1 (T_1 space)**.** Let X be a topological space. We say that X is T_1 or Fréchet if

$$\text{for any distinct } x, y \in X, \quad \text{there exists an open set } U \subseteq X \quad \text{such that } x \in U \text{ but } y \notin U.$$

Definition 2.2 (T_2 space)**.** Let X be a topological space. We say that X is T_2 or Hausdorff if

$$\begin{aligned} \text{for any distinct } x, y \in X, \quad & \text{there exist open neighbourhoods } U \text{ and } V \text{ of } x, y \text{ respectively} \\ & \text{such that } U \cap V = \emptyset \end{aligned}$$

Remark 2.1. Any T_2 space is also T_1.

Example 2.1 (metric spaces are Hausdorff)**.** Any metric space is Hausdorff. To see why, for any $x, y \in X$, let $a = d_X(x, y)$. Define the open neighbourhoods U_x and V_y as follows:

$$U_x = B_{a/3}(x) \text{ and } V_y = B_{a/3}(y) \quad \text{so} \quad U_x \cap V_y = \emptyset.$$

As mentioned, U_x and V_y are two disjoint open neighbourhoods that contain x and y respectively.

Example 2.2 (trivial topology)**.** Let X be a topological space such that $|X| \geq 2$. Then, the trivial topology is not T_1. To see why, consider a topological space X with at least two distinct points, say x and y. In the trivial topology, the only open sets are $\emptyset$ and X, so

$$\text{the only non-empty open set is} \quad X \text{ itself.}$$

So, regardless of which points we pick, the only open set containing x is X, which must also contain y, and vice versa. As such, there are no open sets that can separate distinct points x and y in the sense required by the T_1 property.

Example 2.3 (discrete topology is Hausdorff)**.** The discrete topology is Hausdorff. To see why, consider any two distinct points $x, y \in X$. Recall that in the discrete topology, every subset of X is open, so the singleton sets $\{x\}$ and $\{y\}$ are open. As these singleton sets form neighbourhoods around x and y neighbourhoods, it follows that the neighbourhoods are disjoint.

Example 2.4 (co-finite topology)**.** The co-finite topology is T_1; it is Hausdorff if and only if X is finite.

We first deduce the first claim. Suppose X is a topological space equipped with the co-finite topology. If $x, y \in X$ are distinct, then $X \setminus \{x\}$ is open and thus contains y but not x. This shows that X is T_1.

Now, if X is finite, then the co-finite topology on X is discrete and hence Hausdorff. To see why, recall that the co-finite topology allows all sets with finite complements to be open, so every subset of X is open. This is precisely the definition of the discrete topology, where every subset of X is an open set.

On the other hand, suppose X is infinite. We shall prove that X is not Hausdorff. Suppose $U, V \subseteq X$ are open. Then, there exist

$$x_1, \ldots, x_n, y_1, \ldots, y_n \in X \quad \text{such that} \quad U = X \setminus \{x_1, \ldots, x_n\} \text{ and } V = X \setminus \{y_1, \ldots, y_n\}.$$

Note that

$$X \setminus U = \{x_1, \ldots, x_n\} \text{ and } X \setminus V = \{y_1, \ldots, y_n\} \quad \text{are finite sets.}$$

Since X is infinite, then there exists

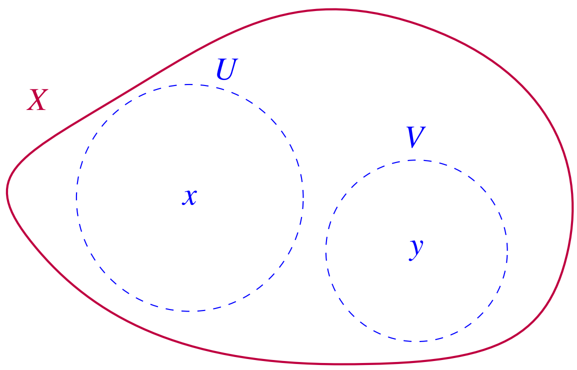
$$z \in X \setminus \{x_1, \ldots, x_n, y_1, \ldots, y_n\} \quad \text{which implies} \quad z \in U \cap V.$$

Hence, X is not Hausdorff since $U \cap V \neq \emptyset$.

Proposition 2.1 (product of Hausdorff spaces)**.** If

$$X \text{ and } Y \text{ are Hausdorff spaces,} \quad \text{then} \quad X \times Y \text{ is also Hausdorff.}$$

Proof. Let X and Y be Hausdorff spaces. Recall that X is Hausdorff if for any distinct points $x, y \in X$, we can construct open balls U and V around x and y respectively such that $y \notin U$ and $x \notin V$.



Similarly, let Y be a Hausdorff space. Then, for any distinct points $x', y' \in Y$, we can construct open balls U' and V' around x' and y' respectively such that $y \notin U'$ and $x \notin V'$.

Now, consider the product of these two Hausdorff spaces, denoted by $X \times Y$. Let $(x, x'), (y, y') \in X \times Y$ be distinct points. Let U'' and V'' be open balls around (x, x') and (y, y') respectively. Suppose

$$r_{U''} = \min\{r_U, r_{U'}\} \quad \text{and} \quad r_{V''} = \min\{r_V, r_{V'}\}$$

which denote the radii of U'' and V'' respectively. Then, by symmetry, it suffices to show that (y,y') is not contained in the open ball U'' containing (x,x'). This is equivalent to

$$\sqrt{(x-y)^2+(x'-y')^2} > \min\{r_U, r_{U'}\} \quad \text{so} \quad (x-y)^2+\left(x'-y'\right)^2 > r_U^2 \text{ and } r_{U'}^2.$$

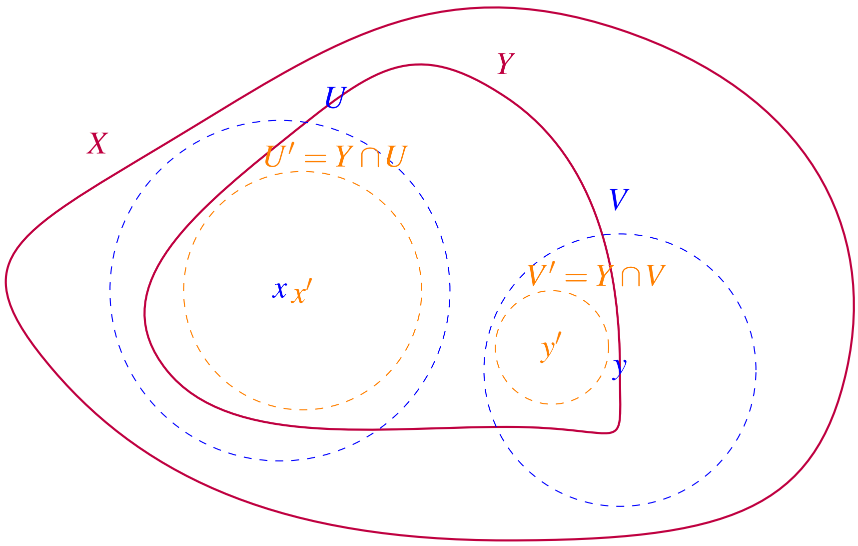
Recall that $|x-y| > r_U$ and $|x'-y'| > r_{U'}$, so the result follows. □

Proposition 2.2 (subspace of Hausdorff space)**.** For any

$$\text{Hausdorff space } X, \quad \text{any subspace of } X \text{ is also Hausdorff.}$$

Proof. Let X be a Hausdorff space. Then, for all distinct points $x, y \in X$, we can construct two open balls U and V centred at x and y respectively such that $x \notin V$ and $y \notin U$. Suppose Y is a subspace of X. Then, consider two points $x', y' \in Y$. Consider the sets

$$U' = Y \cap U \quad \text{and} \quad V' = Y \cap V \quad \text{which are open in the subspace topology on } Y.$$



So,

$$U' \cap V' = U \cap V \cap Y \subseteq U \cap V = \emptyset.$$

So, there exists an open set $U' \subseteq Y$ such that if $x' \in U'$, then $x' \notin V'$. The same symmetric argument holds for y', i.e. there exists an open set $V' \subseteq Y$ such that if $y' \in V'$, then $y' \notin U'$. We conclude that Y is also Hausdorff. □

Proposition 2.3. Suppose X is an infinite set. Then,

$$\text{the cofinite topology on } X \quad \text{is not metrizable.}$$

Example 2.5. In relation to Proposition 2.3, we give an example of a finite set X equipped with the co-finite topology such that X is metrizable.

Let $X = a, b, c\}$ be a finite set with three elements. Then, X is metrizable, i.e. we can define a discrete metric $d : X \times X \to \mathbb{R}$, which is a simple way to metrize the space. The discrete metric on X would be

$$d(a,b) = 1 \quad d(a,c) = 1 \quad d(b,c) = 1 \quad d(x,x) = 0 \text{ for all } x \in X.$$

This metric induces the discrete topology on X, which coincides with the co-finite topology because X is a finite set.

Proposition 2.4. X is a T_1 topological space if and only if for all $x \in X$, $\{x\}$ is closed.

Proof. We first prove the forward direction. It suffices to prove that $X \setminus \{x\}$ is open in X. Let $x \in X$ and for all $y \in X \setminus \{x\}$, there exists an open set $V_y \subseteq X$ such that $y \in V_y$ and $x \notin V_y$. Hence,

$$X \setminus \{x\} = \bigcup_{y \in Y} V_y \quad \text{which is the union of open sets} \quad \text{which is open.}$$

Now, we prove the reverse direction. Let $x, y \in X$ be distinct. Then,

$$U = X \setminus \{x\} \subseteq X \text{ is open} \quad \text{with } x \notin U \text{ and } y \in U.$$

Hence, X is T_1. □

As mentioned in Example 2.1, metric spaces are Hausdorff, so by Remark 2.1, every metric space is also in T_1. As such, we have the following corollary, which appears to be obvious.

Corollary 2.1. Finite sets in metric spaces are closed.

2.2. *First Countable Spaces*

Definition 2.3 (countable basis). Let X be a topological space. For all $x \in X$, a countable basis of X at x is

$$\text{a countable collection } \mathcal{B} \quad \text{of open sets in } X \text{ that contain } x \quad \text{such that}$$
$$\text{every open set in } X \text{ that contains } x \quad \text{also contains some } B \in \mathcal{B}.$$

Definition 2.4 (first countable space). A topological space X is said to be first countable if there exists a countable basis of X at x for every $x \in X$.

Example 2.6 (metric spaces are first countable). Metric spaces are first countable. To see why, let $x \in X$ be arbitrary. Then, consider the following set $\mathcal{B}$ of open balls:

$$\mathcal{B} = \left\{ B_{1/i}(x) : i \in \mathbb{Z}^+ \right\} \quad \text{which is a countable basis of } X \text{ at } x$$

To see why this holds, note that the sequence $1/i$ converges to 0 as $i \to \infty$, so there exists $i \in \mathbb{Z}^+$ such that $1/i < \varepsilon$, implying that $B_{1/i}(x) \subseteq B_\varepsilon(x) \subseteq U$, where U is any open set containing x.

Example 2.7 (co-finite topology on uncountable set is not first countable)**.** The co-finite topology on an uncountable set X is not first countable. For example, take $X = \mathbb{R}$.

To see why, pick $x \in X$ and suppose on the contrary that

$$\text{there exists} \quad \text{a countable basis } \mathcal{B} = \{B_1, \ldots, \} \text{ at } x.$$

Then, $B_i = X \backslash F_i$ for some finite $F_i \subseteq X$. Since X is uncountable, then

$$\text{there exists } y \in X \backslash \left(\{x\} \cup \bigcup \bigcup_{i=1}^{\infty} F_i \right) \quad \text{and let} \quad U = X \backslash \{y\}.$$

Note that $x \in U$, however for all i, we have $y \in B_i$ but $y \notin U$, so B_i is not a subset of U for every i. We have arrived at a contradiction so we conclude that $\mathcal{B}$ is uncountable.

Proposition 2.5. Let X be a topological space.

(i) Let $A \subseteq X$. If there exists a sequence $\{x_i\}_{i=1}^{\infty}$ such that $x_i \to x$ as $i \to \infty$, then $x \in \overline{A}$. The converse is true if X is first countable.

(ii) Let $f : X \to Y$. If f is continuous, then for any sequence $\{x_i\}_{i=1}^{\infty} \subseteq X$ such that $x_i \to x$ as $n \to \infty$, we have $f(x_i) \to f(x)$ as $i \to \infty$. The converse holds if X is first countable.

Proof. We first prove **(i)**. We start with the forward direction. Let $\{x_i\}_{i=1}$ be a sequence such that $x_i \to x$ as $i \to \infty$. Recall that $\overline{A}$ is the union of A and the set of limit points of A. As such, it suffices to prove that if $x \notin A$, then $x_i \neq x$ for every i.

Given that $x_i \to x$, then for all open $U \subseteq X$ such that $x \in U$, there exists $N_U > 0$ such that $x_i \in U$ for all $i \geq N_U$. So, there exists $x_{N_U} \in \{x_i\}_{i=1}^{\infty}$ such that $x_{N_U} \in U \cap (A \backslash \{x\})$. Hence, x is a limit point of A.

We now prove the reverse direction. Suppose $x \in \overline{A}$. Let $\mathcal{B} = \{B_1, \ldots\}$ be a countable basis of X at x. Without loss of generality, we may assume that $B_i \supseteq Bi + 1$ for all $i \in \mathbb{Z}^+$. For any i, choose

$$x_i \in \bigcap_{j=1}^{i} B_j = A = B_i \cap A.$$

Hence, for any open $U \subseteq X$ that contains x, there exists $N \in \mathbb{Z}^+$ such that $B_N \subseteq U$. So, for all $i \geq N$, we have $x_i \in B_N \subseteq U$, implying $x_i \to x$.

As for **(ii)**, we first prove the forward direction. Let $U \subseteq Y$ be an open set such that $f(x) \in U$. Then, $f^{-1}(U) \subseteq X$ is open and $x \in f^{-1}(U)$. If $x_i \to x$, then there exists $N > 0$ such that $x_i \in f^{-1}(U)$ for all $i \geq N$. Hence, $f(x_i) \in U$ for all $i \geq N$. Since U was an arbitrary open set, the result follows. We omit the proof of the reverse direction. □

2.3. *Compactness*

Definition 2.5 (open cover). Let X be a topological space. An open cover of X is a collection of open sets $\{U_\alpha\}_{\alpha\in\Lambda}$ in X such that

$$\bigcup_{\alpha\in\Lambda} U_\alpha = X.$$

Definition 2.6. A topological space X is

compact if every open cover of X admits a finite subcover.

Example 2.8. We have

$$X = \left\{\frac{1}{n} : n \in \mathbb{Z}^+\right\} \subseteq \mathbb{R} \quad \text{not being compact}$$

since the set of positive real numbers of the form $1/n$ forms an open cover of X with respect to the subspace topology but it does not have a finite subcover.

Example 2.9 (Rudin PMA p. 44 Question 10). Let S be the set consisting of 0 and all real numbers of the form $1/n$, where $n \in \mathbb{N}$. Prove that S is compact.

Solution. Suppose on the contrary that S is not compact. That is, there exists an open cover $\mathcal{U}$ of S which does not contain a finite subcover. Consider partitioning the interval $[0,1]$ into the following two sets, A_1 and B_1:

$$A_1 = \left\{s \in S : s \in \left[0, \frac{1}{2}\right]\right\} \quad \text{and} \quad B_1 = \left\{s \in S : s \in \left[\frac{1}{2}, 1\right]\right\}$$

Note that B_1 is finite since it contains the elements $1/2$ and 1. Hence, B_1 has a finite subcover. However, A_1 does not have a finite subcover because $A_1 \cup B_1 = S$, so it is not compact.

Now, consider the partition

$$A_2 = \left\{s \in S : s \in \left[0, \frac{1}{4}\right]\right\} \quad \text{and} \quad B_2 = \left\{s \in S \mid s \in \left[\frac{1}{4}, \frac{1}{2}\right]\right\}.$$

In a similar fashion, B_2 is compact but A_2 is not compact. Define A_n and B_n for general $n \in \mathbb{N}$ as follows:

$$A_n = \left\{s \in S : s \in \left[0, \frac{1}{2^n}\right]\right\} \quad \text{and} \quad B_n = \left\{s \in S : s \in \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]\right\}$$

Observe that $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ and all the B_n's are compact sets. As mentioned, as we cannot obtain finite subcover for A_1 and A_2, in general, we cannot obtain finite subcover for A_n for $n \in \mathbb{N}$. Suppose $1/r \in A_n$. Then, $1/r \leq 1/2^n$, so $r \geq 2^n$. Also, as $\mathcal{U}$ is open, then for all $x \in \mathbb{R}$, there exists $R \in \mathbb{R}^+$ such that $(x - R, x + R) \subseteq \mathcal{U}$.

Let $1/r \in A_n$, so for sufficiently large n,

$$\left(\frac{1}{2^n} - R, \frac{1}{2^n} + R\right) \in \mathcal{U}.$$

Now, note that

$$\left[0, \frac{1}{2^n}\right] \subseteq \left(\frac{1}{2^n} - R, \frac{1}{2^n} + R\right) \subseteq \mathcal{U},$$

where the interval on the left comprises all s in A_n. As such, $A_n \subseteq \mathcal{U}$. This is a contradiction since A_n has a finite subcover. □

Example 2.10. Any metric space X of infinite diameter is not compact. Let $x \in X$. Then, $\{B_n(x) : n \in \mathbb{Z}^+\}$ forms an open cover of X which does not have a finite subcover.

Example 2.11 (MA3209 AY24/25 Sem 1 Homework 4)**.** Suppose X and Y are topological spaces. Let $f : X \to Y$ be a continuous map. Prove that

$$\text{if } X \text{ is compact,} \quad \text{then} \quad f(X) \text{ is compact.}$$

Solution. Suppose X is compact and let $\{V_\alpha\}$ be an open cover of $f(X)$ in Y. We wish to prove that $f(X)$ has a finite subcover. Since f is continuous, the preimages of open sets in Y are open in X. Hence, for each open set V_α in the open cover of $f(X)$, the preimage $f^{-1}(V_\alpha)$ is open in X.

We see that the collection $\{f^{-1}(V_\alpha)\}$ forms an open cover of X because

$$X = f^{-1}(f(X)) = \subseteq_{\alpha \in \Lambda} f^{-1}(V_\alpha).$$

Since X is compact, there exists a finite subcover that covers X, i.e.

$$X = \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i}).$$

Applying f to both sides, we have

$$f(X) = f\left(\bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})\right) \subseteq \bigcup_{i=1}^{n} f\left(f^{-1}(V_{\alpha_i})\right) \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$$

which shows that $f(X)$ has a finite subcover. □

Remark 2.2. Let X be a topological space. $Y \subseteq X$ is a compact subspace if and only if every collection $\mathcal{U}$ of open sets in Y such that

$$Y \subseteq \bigcup_{U \in \mathcal{U}} U \quad \text{admits a finite subcollection } \mathcal{U}' \subseteq \mathcal{U}$$

such that

$$Y \subseteq \bigcup_{U \in \mathcal{U}'} U.$$

Proposition 2.6. Every closed subspace of a compact space is compact.

Proof. Let X be a compact space and $Y \subseteq X$ is closed. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a cover of Y by open sets in X. Notice that

$$\{U_\alpha\}_{\alpha\in\Lambda} \cup \{X \setminus Y\} \quad \text{is an open cover of } X.$$

Also, the compactness of X guarantees that there exists a finite subcover

$$\mathcal{U}'' \subseteq \{U_\alpha\}_{\alpha\in\Lambda} \cup \{X \setminus Y\} \text{ of } X.$$

If $\mathcal{U}' \subseteq \{U_\alpha\}_{\alpha\in\Lambda}$, then this is a finite subcover of Y. On the other hand, if $X \setminus Y \in \mathcal{U}'$, then $U' \setminus \{X \setminus Y\} \subseteq \{U_\alpha\}_{\alpha\in\Lambda}$ is a finite subcover of Y. □

Proposition 2.7. Every compact subspace of a Hausdorff space is closed.

Proposition 2.8 (tube lemma)**.** Let X be a topological space and Y is a compact topological space. If $N \subseteq X \times Y$ is an open set that contains $\{(x_0, y) : y \in Y\}$ (i.e. the *tube* around the point $x_0 \in X$), then N contains $W \times Y$ for some open $W \subseteq X$ that contains x_0.

Example 2.12. Let

$$S = \left\{ (x,y) \in \mathbb{R}^2 : |x| \leq \frac{1}{y^2+1} \right\} \subseteq \mathbb{R}^2$$

which contains $\{0\} \times \mathbb{R}$, but it does not contain a tube. To see why, there does not exist any open set $(-\varepsilon, \varepsilon) \times \mathbb{R}$, where $\varepsilon > 0$, which is contained in S. This fact is easy to establish because for y sufficiently large enough, we have $1/(y^2+1) < \varepsilon$, so the width of the strip around $x = 0$ becomes very narrow, to the extent that it is narrower than $(-\varepsilon, \varepsilon)$.

Corollary 2.2. If X and Y are compact topological spaces, then $X \times Y$ is compact.

We will see a generalisation of Corollary 2.2 in Theorem 4.2, known as Tychonoff's theorem. Assuming the axiom of choice, it states that the arbitrary product of compact spaces is also compact.

Definition 2.7 (finite intersection property)**.** A collection $\mathcal{G}$ of subsets of X has finite intersection property if

$$\text{every finite subcollection } \{G_1, \ldots, G_n\} \subseteq \mathcal{G} \quad \text{satisfies } \bigcap_{i=1}^{n} G_i \neq \emptyset.$$

Proposition 2.9. Saying that a topological space X is compact is equivalent to X having the following property: for a collection $\mathcal{G}$ of closed sets in X, if $\mathcal{G}$ has the finite intersection property, then

$$\bigcap_{G \in \mathcal{G}} G \neq \emptyset.$$

Proof. Let $\mathcal{G}$ be a collection of closed sets in X. Define $\mathcal{U}$ to be the following set:

$$\mathcal{U} = \{X \backslash G : G \in \mathcal{G}\}$$

Then, the following statements are equivalent:

(i) If

$$\mathcal{G} \text{ has the finite intersection property}, \quad \text{then } \bigcap_{G \in \mathcal{G}} G \neq \emptyset$$

(ii) If

$$\bigcap_{G \in \mathcal{G}} G = \emptyset, \quad \text{then there exist } G_1, \ldots, G_n \in \mathcal{G} \quad \text{such that } \bigcap_{i=1}^{n} G_i = \emptyset$$

(iii) If

$$\bigcup_{U \in \mathcal{U}} U = X, \quad \text{then there exist } U_1, \ldots, U_n \in \mathcal{U} \quad \text{such that } \bigcup_{i=1}^{n} U_i = X$$

□

Corollary 2.3. If X is compact and $\{G_i\}_{i=1}^{\infty}$ is a nested sequence of closed subsets of X (i.e. $G_i \supseteq G_{i+1}$ for all $i \in \mathbb{Z}^+$), then

$$\bigcap_{i=1}^{\infty} G_i \neq \emptyset.$$

Definition 2.8 (isolated point). A point x in a topological space X is isolated if $\{x\}$ is open in X.

Theorem 2.1. Let X be a non-empty, compact, Hausdorff space. If X has no isolated points, then X is uncountable.

Lemma 2.1. Let X be a Hausdorff space. If $U \subseteq X$ is non-empty and open and $x \in X$ is not an isolated point, then there exists an open, non-empty $V \subseteq U$ such that $x \notin \overline{V}$.

2.4. *Limit Points and Sequential Compactness*

Definition 2.9 (limit point compactness)**.** A topological space X is limit point compact if every infinite subset of X has a limit point in X.

Example 2.13 (unbounded metric space is not limit point compact)**.** Any unbounded metric space (X,d) is not limit point compact. For example, the Euclidean space $\mathbb{R}^n$ is unbounded. Say we equip $\mathbb{R}^n$ with the standard distance metric $d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|$. Then, d is unbounded because for any two points x and y, the distance can get arbitrarily large.

Pick a point $x_1 \in X$. For all $i \in \mathbb{Z}^+$, let x_i be any point satisfying $B_i(x_1)\backslash B_{i-1}(x_1)$. We will show that the infinite set $S = \{x_1, x_2, \ldots\} \subseteq X$ does not have any limit points.

We claim that if $j \neq i, i-1, i+1$, then $d(x_j, x_i) > 1$. Suppose on the contrary that

$$\text{there exists } j \neq i, i-1, i+1 \quad \text{such that} \quad d(x_j, x_i) \leq 1.$$

If $j > i+1$, then

$$\begin{aligned} d(x_j,x_1) &\leq d(x_j,x_i) + d(x_i,x_1) \quad \text{by triangle inequality} \\ &\leq 1+i \end{aligned}$$

To see why the second inequality holds, we have $d(x_j,x_i) \leq 1$ by assumption and $d(x_i,x_1) = i$ since $x_i \in B_i(x_1)\backslash B_{i-1}(x_1)$ so x_i is of distance at most i from x_1. This is a contradiction since $x_j \notin B_{j-1}(x_1) \supseteq B_{i+1}(x_1)$.

Similarly, if $j < i-1$, then one is able to deduce that $d(x_i,x_1) \leq 1+j$, which is a contradiction since $x_i \notin B_{i-1}(x_1) \supseteq B_{j+1}(x_1)$.

Choose any $y \in X\backslash A$, then we deduce that $\left|B_{1/2}(y) \cap A\right| \leq 3$. By the definition of y, we have $y \notin B_{1/2}(y) \cap A$, so there exists $\varepsilon > 0$ such that $B_\varepsilon(y) \cap A = \emptyset$. This concludes that y is not a limit point of A.

Example 2.14. It is possible for a topological space to be limit point compact but not compact. For example, consider the space $[0,1)$ (i.e. the half-open interval) with the usual topology inherited from $\mathbb{R}$.

Note that any infinite subset of $[0,1)$ will accumulate near 1 since there is no point beyond 1 to escape to. Hence, every infinite subset will have a limit point in $[0,1)$, making the space limit point compact. However, recall from MA2108 that $[0,1)$ is not compact because the open cover consisting of intervals $(1/n,1)$ for $n = 1,2,\ldots$ covers $[0,1)$, but no finite subcover can cover the entire space.

Proposition 2.10. If X is compact, then it is limit point compact.

Definition 2.10 (sequential compactness)**.** Let X be a topological space. X is sequentially compact if every sequence in X has a convergent subsequence.

Proposition 2.11. If X is sequentially compact, then X is limit point compact.

Proof. Let A be an infinite subset of X. So, we can construct a sequence $\{x_n\}$ of points in A. By sequential compactness of X, the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ with limit $x \in X$.

It suffices to show that x is a limit point of A. Suppose on the contrary that x is not a limit point of A. Then, there exists an open neighbourhood U of x such that $U \cap (A \setminus \{x\}) = \emptyset$. However, since $x_{n_k} \to x$, we must have infinitely many terms of x_{n_k} lie in U, contradicting the earlier statement that $U \cap (A \setminus \{x\}) = \emptyset$. So, x must be a limit point of A. □

Definition 2.11 (Lebesgue number)**.** Let X be a metric space and let $\mathcal{U}$ be an open cover of X. $\delta > 0$ is a Lebesgue number for $\mathcal{U}$ if for all subsets $S \subseteq X$ such that $\operatorname{diam}(S) < \delta$, there exists $U \in \mathcal{U}$ such that $S \subseteq U$.

Example 2.15 (real line with open intervals)**.** Let $X = [0,1] \subseteq \mathbb{R}$ be endowed with the usual Euclidean metric, and consider the open cover

$$\mathcal{U} = \left\{ \left(-\frac{1}{2}, 1\right), \left(0, \frac{3}{4}\right), \left(\frac{1}{4}, \frac{3}{2}\right) \right\}.$$

For this cover of X, we need to find a number $\delta > 0$ such that any subset $S \subseteq [0,1]$ with $\operatorname{diam}(S) < \delta$ is fully contained in at least one set in $\mathcal{U}$.

Observe that the interval $[0,1]$ is fully contained within the union of the sets in $\mathcal{U}$. We see that $\delta = 1/4$ works as a Lebesgue number for $\mathcal{U}$ since any subset of $[0,1]$ with diameter less than $1/4$ will lie within one of the intervals in $\mathcal{U}$. In fact, choosing any $0 < \delta < 1/4$ works too.

Example 2.16 (circle with open arcs)**.** Let $X = \mathbb{S}^1$ to be the unit circle with the metric induced by the Euclidean distance in $\mathbb{R}^2$. Consider an open cover of $\mathbb{S}^1$ made up of three open arcs $\mathcal{U} = \{U_1, U_2, U_3\}$, where each U_i is an open arc covering roughly one third of the circle but with overlap. From this cover, we can choose a Lebesgue number δ to be the length of the shortest arc among U_1, U_2, U_3. Here, since each open set overlaps with the others enough to cover any subset of $\mathbb{S}^1$ with sufficiently small diameter, δ can be chosen to be approximately one-third of the circumference of $\mathbb{S}^1$.

Example 2.17 (finite discrete space)**.** Let $X = \{1,2,3\}$ be a discrete metric space endowed with the discrete metric. So, $d(x,y) = 1$ if x and y are distinct; $d(x,y) = 0$ otherwise. Let $\mathcal{U} = \{\{1\}, \{2\}, \{3\}\}$ be an open cover of X. Then, any $\delta \leq 1$ works as a Lebesgue number for $\mathcal{U}$ since for any subset

$S \subseteq X = \{1,2,3\}$ with $\operatorname{diam}(S) < 1$ must consist of a single point, which will be contained in one of the open sets in $\mathcal{U}$.

Lemma 2.2. If X is a sequentially compact metric space, then every open cover of X has a Lebesgue number.

Definition 2.12 (totally bounded)**.** A metric space X is totally bounded if

$$\text{for all } \varepsilon > 0 \quad \text{there exists a finite subcover of } X \text{ by balls of radius } \varepsilon.$$

Example 2.18 (closed interval in $\mathbb{R}$)**.** Let $X = [0,1] \subseteq \mathbb{R}$ be equipped with the standard Euclidean metric $d(x,y) = |x-y|$. To check total boundedness, take any $\varepsilon > 0$. Since X is bounded, we can cover $[0,1]$ by a finite number of open intervals (balls as a generalisation) of radius ε. Specifically, we can partition $[0,1]$ into intervals of length ε or less and place a ball of radius ε at each endpoint of these intervals.

Example 2.19 (closed balls in $\mathbb{R}^n$)**.** Naturally, we can extend Example 2.18 to closed balls in $\mathbb{R}^n$. Let $X = \overline{B_0(R)}$ to be the closed ball of radius R centred at the origin in $\mathbb{R}^n$ equipped with the Euclidean metric, i.e.

$$X = \overline{B_0(R)} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq R\}.$$

It is easy to show that X is totally bounded, i.e. cover $\overline{B_0(R)}$ by a finite number of balls of radius ε. One way to do this is by constructing a lattice of points within the ball where each point is separated by less than ε. The compactness of the closed ball ensures that we only need a finite number of these lattice points to cover $\overline{B_0(R)}$ with balls of radius ε.

Lemma 2.3. If X is sequentially compact and metrizable, then X is totally bounded.

Proof. Suppose on the contrary that X is not totally bounded, i.e. there exists $\varepsilon_0 > 0$ such that X does not admit a finite cover by ε_0-balls. Consider the following sequence:

$$\begin{aligned} &x_1 \in X \\ &x_2 \in X \backslash B_{\varepsilon_0}(x_1) \\ &x_3 \in \backslash \left(B_{\varepsilon_0}(x_1) \cup B_{\varepsilon_0}(x_2)\right) \end{aligned}$$

so in general, we have

$$x_n \in X \backslash \left(\bigcup_{i=1}^{n-1} B_{\varepsilon_0}(x_i) \right).$$

All these give a sequence $\{x_n\}_{n=1}^{\infty}$ in X. Note that if $i \neq j$, then $d(x_i, x_j) \geq \varepsilon$, so for all $x \in X$, $B_{\varepsilon/2}(x)$ contains at most one x_i. As such, $\{x_n\}_{n=1}^{\infty}$ has no convergent subsequence, contradicting that X is not sequentially compact. □

Theorem 2.2. If X is metrizable, then the following are equivalent:

(i) X is compact

(ii) X is limit point compact

(iii) X is sequentially compact

Corollary 2.4. Let

$$f : (X, d_X) \to (Y, d_Y) \quad \text{be continuous.}$$

If X is compact, then f is uniformly continuous.

2.5. *Complete and Totally Bounded Metric Spaces*

Proposition 2.12. If X is totally bounded, then X has finite diameter.

Example 2.20. $\mathbb{R}^n$ equipped with the L^p metric has infinite diameter, so it is not totally bounded.

Example 2.21. Let (X, d) be a metric space and define

$$\rho(x,y) = \frac{d(x,y)}{1 + d(x,y)}.$$

Prove that (X, ρ) is totally bounded if and only if (X, d) is totally bounded.

Proof. We first prove the forward direction. Suppose for every $\varepsilon > 0$, there exists a finite cover of X by open ρ-balls of radius ε. Let $\delta > 0$ be arbitrary. Then, as

$$d(x,y) = \frac{\rho(x,y)}{1 - \rho(x,y)},$$

for any $\delta > 0$, there exists $\varepsilon > 0$ such that $\delta = \varepsilon / (1 - \varepsilon)$, which is equivalent to saying that $\varepsilon = \delta / (1 + \delta)$. If $\rho(x,y) < \varepsilon$, then

$$d(x,y) < \frac{\varepsilon}{1 - \varepsilon} = \delta$$

which shows that (X, d) is totally bounded.

We then prove the reverse direction. Suppose $\delta > 0$ is arbitrary. Then, there exists $\varepsilon > 0$ such that $\varepsilon = \delta$. So,

$$d(x,y) < \delta \quad \text{implies} \quad \rho(x,y) = \frac{d(x,y)}{1 + d(x,y)} < d(x,y) < \delta = \varepsilon$$

which shows that any open d-ball of radius δ around a point x is contained within an open p-ball of radius ε around the same point x. This shows that (X, ρ) is totally bounded. □

Definition 2.13 (Cauchy sequence and convergence)**.** Let (X,d) be a metric space. A sequence of points $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence if for all $\varepsilon > 0$, there exists $N > 0$ such hat $d(x_n, x_m) < \varepsilon$ for all $m,n > N$. A metric space is complete if every Cauchy sequence converges.

Example 2.22. $\mathbb{R}^n$ with respect to the L^p metric for $p \in [1,\infty]$ is complete.

Example 2.23. Equipped with the standard metric on $\mathbb{R}$ restricted to $\mathbb{Q}$, we see that $\mathbb{Q}$ is not complete. Since $\mathbb{Q} \subseteq \mathbb{R}$ is dense, there exists a sequence in $\mathbb{Q}$ that converges in $\mathbb{R}$ to an irrational number. Such sequences are Cauchy but do not have a convergent subsequence in $\mathbb{Q}$.

Example 2.24. Let d be the standard metric on $\mathbb{R}$ and ρ be the metric on $\mathbb{R}$ given by

$$\rho(x,y) = \frac{d(x,y)}{1+d(x,y)}.$$

Also, D is the metric on $\prod_{\mathbb{Z}} \mathbb{R} = R^{\omega}$ given by

$$D(x,y) = \sup\left\{\frac{\rho(\pi_k(x),\pi_k(y))}{k} : k \in \mathbb{Z}^+\right\} \quad \text{where } \pi_k : \mathbb{R}^{\omega} \to \mathbb{R} \text{ is the projection to the } k^{\text{th}} \text{ factor.}$$

Recall that the topology on $\mathbb{R}^{\omega}$ induced by D is the product topology. We note that $(\mathbb{R}^{\omega}, D)$ is a complete metric space, which follows from the fact that $(\mathbb{R},\rho)$ is complete.

Theorem 2.3. A metric space (X,d) is compact if and only if it is complete and totally bounded.

Corollary 2.5 (Heine-Borel theorem)**.** A subspace $G \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

2.6. *Local Compactness*

Definition 2.14 (local compactness)**.** A topological space is locally compact at $x \in X$ if there exists a compact set $C \subseteq X$ and an open set $U \subseteq X$ such that $x \in U \subseteq C$. If X is locally compact at every $x \in X$, then X is locally compact.

Example 2.25. $X = \mathbb{R}^n$ is locally compact. To see why, for every $\mathbf{x} \in X$, $\overline{B_\varepsilon(\mathbf{x})}$ is closed and bounded, hence compact. Take

$$U = B_\varepsilon(\mathbf{x}) \text{ which is open} \quad \text{and} \quad C = \overline{B_\varepsilon(\mathbf{x})} \text{ which is compact.}$$

Note that $\mathbf{x} \in U \subseteq C$. Since $\mathbb{R}^n$ is locally compact at every $x \in \mathbb{R}^n$, then X is locally compact.

Example 2.26. $\mathbb{Q} \subseteq \mathbb{R}$ is not locally compact. To see why, let $U \subseteq \mathbb{Q}$ be an open set. It suffices to show that any $C \subseteq \mathbb{Q}$ that contains U is not compact. We shall use the following facts:

$$\mathbb{Q} \text{ is dense} \quad \text{and} \quad \mathbb{R}\backslash\mathbb{Q} \text{ is dense.}$$

Since $U \subseteq \mathbb{Q}$ is open, there exist $a, b \in \mathbb{R}$ such that $a < b$ and $(a,b) \cap \mathbb{Q} \subseteq U$. So, there exists a sequence $\{x_i\}_{i=1}^{\infty}$ in U such that $x_i \to p \in (a,b) \in \mathbb{R} \backslash \mathbb{Q}$. So, any subsequence of x_i converges to p in $\mathbb{R}$, implying that no subsequence of x_i converges in $\mathbb{Q}$.

As such, if $C \subseteq \mathbb{Q}$ contains U, then x_i lies in C, but does not have a convergent subsequence in C. Hence, C is not compact.

Example 2.27. $\mathbb{R}^{\omega}$ equipped with the product topology is not locally compact. Let $U \subseteq \mathbb{R}^{\omega}$ be open. In a similar fashion, it suffices to show that if $C \subseteq \mathbb{R}^{\omega}$ is a set that contains U, then C is not compact.

Suppose on the contrary that there exists $C \subseteq \mathbb{R}^{\omega}$ compact such that $U \subseteq C$. Recall that $\mathbb{R}$ is a Hausdorff space, so $\mathbb{R}^{\omega}$ is also Hausdorff, which shows that C is closed in $\mathbb{R}^{\omega}$.

Since U is open, there exists a finite subset of $\mathbb{Z}^+$ and $a_i < b_i$ for every $i \in \Lambda$ such that

$$B = \prod_{i \in \Lambda} (a_i, b_i) * \prod_{i \in \mathbb{Z}^+ \backslash \Lambda} \mathbb{R} \subseteq U \subseteq C.$$

This shows that

$$\overline{B} = \prod_{i \in \Lambda} [a_i, b_i] * \prod_{i \in \mathbb{Z}^+ \backslash \Lambda} \mathbb{R} \subseteq C \quad \text{implying that} \quad \overline{B} \text{ is compact.}$$

However, $\overline{B}$ is not sequentially compact as the sequence $\{x_i\}_{i=1}^{\infty}$ in $\overline{B}$ given by $\pi_1(x_n) = a_i$ for all $i \in \Lambda$ and $\pi_i(x_n) = n$ for all $i \in \mathbb{Z}^+ \backslash \Lambda$ has no convergent subsequence.

Theorem 2.4. Let X be a topological space. X is locally compact and Hausdorff if and only if there exists a compact Hausdorff space Y and a map $h_Y : X \to Y$ such that

$$h_Y \text{ is a homeomorphism onto its image} \quad \text{and} \quad Y \backslash h_Y(X) \text{ is a single point.}$$

Definition 2.15 (compactification and Alexandroff compactification). Suppose Y is a compact, Hausdorff space, and there exists a map $h : X \to Y$ such that $\overline{h(X)} = Y$, and h is a homeomorphism onto its image, then Y is a compactification of X.

Furthermore, if $Y \backslash h_Y(X)$ is a point, then Y is the one-point compactification of X (also known as Alexandroff compactification).

Example 2.28 (one-point compactification from $\mathbb{R}^n$ to $\mathbb{S}^n$). Say we add a point at infinity to $\mathbb{R}$. The one-point compactification of $\mathbb{R}$ is homeomorphic to the circle $\mathbb{S}^1$. To see why, $\mathbb{R}$ can be *wrapped around* to meet at the point at infinity, forming a closed loop.

We can generalise this concept to the one-point compactification of $\mathbb{R}^n$. The one-point compactification of $\mathbb{R}^n$ is homeomorphic to the n-dimensionl sphere $\mathbb{S}^n$. This transforms the unbounded Euclidean space into a compact, closed surface.

Example 2.29 (one-point compactification from $\mathbb{C}$ to $\mathbb{C}\cup\{\infty\}$)**.** Say we add a point at infinity to $\mathbb{C}$. The result is a one-point compactification of the complex plane, which is the Riemann sphere. This is a model for the extended complex plane $\mathbb{C}\cup\{\infty\}$ (will see again in MA3211S/MA5217).

Example 2.30. Let

$$\mathbb{D}=\left\{(x,y):x^2+y^2<1\right\} \quad \text{denote the open unit disc.}$$

Then, $\overline{\mathbb{D}}$ and $\mathbb{S}^2$ are compactifications of $\mathbb{D}$. Here, $\overline{\mathbb{D}}$ denotes the closure of $\mathbb{D}$ — it includes all the points of $\mathbb{D}$ along with its boundary, i.e.

$$\overline{\mathbb{D}}=\left\{(x,y):x^2+y^2\leq 1\right\}.$$

$\overline{\mathbb{D}}$ is compact in $\mathbb{R}^2$ as it is both closed (contains all limit points of $\mathbb{D}$) and bounded.

In particular, $\mathbb{S}^2$ is the one-point compactification of $\mathbb{D}$. Here, $\mathbb{S}^2$ denotes the two-dimensional sphere. The idea here is to add a single *point at infinity* to $\mathbb{D}$ to form a compact space. This point at infinity intuitively brings together all the directions along which point in $\mathbb{D}$ could escape if it were left open.

Proposition 2.13. Let X be a Hausdorff topological space. Then, X is locally compact is equivalent to saying that for any $x\in X$, for any open $U\subseteq X$ such that $x\in U$, there exists open $V\subseteq X$ such that

$$x\in V,\overline{V}\subseteq U \quad \text{and} \quad \overline{V} \text{ is compact.}$$

Corollary 2.6. Let X be a locally compact topological space. If

$$A\subseteq X \text{ is closed} \quad \text{or} \quad X \text{ is Hausdorff and } A \text{ is open},$$

then A is locally compact.

Corollary 2.7. X is a homeomorphism to an open subset of a compact Hausdorff space if and only if X is locally compact and Hausdorff.

2.7. *Spaces of Maps and Metric Completion*

Definition 2.16 (uniform metric and uniform topology)**.** Let (Y,d) be a metric space and ρ be the metric on Y given by

$$\rho(x,y)=\frac{d(x,y)}{1+d(x,y)}.$$

The uniform metric on

$$Y^{\Lambda} = \prod_{\alpha\in\Lambda} Y$$

is the metric given by

$$\overline{\rho}(x,y) = \sup\{\rho(\pi_\alpha(x),\pi_\alpha(y)) : \alpha\in\Lambda\}.$$

The uniform topology on Y^{Λ} is the topology generated by the uniform metric.

Remark 2.3. The uniform topology on $\mathbb{R}^{\lambda}$ is finer than the product topology but coarser than the box topology; these three topologies are all different if Λ is infinite.

Proposition 2.14. If (Y,d) is complete, then $\left(Y^{\Lambda},\overline{\rho}\right)$ is complete.

Let X be a topological space and (Y,d) be a metric space. Define

$$\mathcal{C}(X,Y) = \left\{f\in Y^X : f \text{ is continuous}\right\}$$
$$\mathcal{B}(X,Y) = \left\{f\in Y^X : f(X)\subseteq Y \text{ has bounded diameter}\right\}$$

Theorem 2.5. $\mathcal{C}(X,Y),\mathcal{B}(X,Y)\subseteq Y^X$ are closed in the uniform topology. In particular, if (Y,d) is complete, then so are $(\mathcal{C}(X,Y),\overline{\rho})$ and $(\mathcal{B}(X,Y),\overline{\rho})$.

Definition 2.17 (supremum metric)**.** Let $\mathcal{B}(X,Y)$ be the set of functions f in Y^X such that $f(X)\subseteq Y$ has bounded diameter. Define the supremum metric $d_{\sup}$ on $\mathcal{B}(X,Y)$ by

$$d_{\sup}(f,g) = \sup\{d(f(x),g(x)) : x\in X\}.$$

Definition 2.18 (isometric embedding and isometry)**.** Let (X,d_X) and (Y,d_Y) be two metric spaces. We say that

$$f:(X,d_X)\to(Y,d_Y) \quad \text{is an isometric embedding}$$

if

$$d_X(a,b) = d_Y(f(a),f(b)).$$

We say that f is an isometry if it is a surjective isometric embedding.

Definition 2.19 (metric completion)**.** If (X,d_X) is a metric space, then a metric completion of X is a complete metric space (Y,d_Y) and an isometric embedding $\phi: X\to Y$ such that $\overline{\phi(X)} = Y$.

3. Further Properties of Topological Spaces

3.1. *Connectedness in Topological Spaces*

Definition 3.1 (separation and connectedness). Let X be a topological space. A separation of X is a pair U,V of disjoint, non-empty open subsets of X such that $X = U \cup V$. X is connected if there does not exist a separation of X.

Proposition 3.1. X is connected if and only if the only sets in X that are open and closed are $\emptyset$ and X.

Proof. We first prove the forward direction. Suppose X is a connected topological space. Let $U \subseteq X$ be an arbitrary open and closed set. Then, $X \backslash U$ is also open and closed. Since

$$U \cup (X \backslash U) = X \quad \text{and} \quad U \cap (X \backslash U) = \emptyset,$$

it follows that if $U \neq \emptyset$ or X, then $U, X \backslash U$ is a separation of X. Since X is connected, then the only open and closed sets in X are $\emptyset$ and X.

We then prove the reverse direction. Suppose on the contrary that X is not connected. Then, there exist non-empty open sets $U,V \subseteq X$ such that $U \cap V = \emptyset$ and $U \cup V = X$. It follows that $U \neq \emptyset, X$. □

Example 3.1. The trivial topology is connected.

Example 3.2. $[-1,0) \cup (0,1] \subseteq X$ is not connected with respect to the standard topology. This is simply because

$$[-1,0) \cup (0,1] \quad \text{is the disjoint union of } [-1,0) \text{ and } (0,1].$$

Example 3.3. $\mathbb{Q} \subseteq \mathbb{R}$ is not connected with respect to the standard topology.

Example 3.4. $(a,b) \subseteq \mathbb{R}$ is connected. The same claim can be made for $(a,b]$, $[a,b)$, $[a,b]$.

To see why the open interval (a,b) is connected, we shall argue by contradiction. Suppose on the contrary that (a,b) is not connected. Then, there exist non-empty open sets $U,V \subseteq (a,b)$ such that

$$U \cup V = (a,b) \quad \text{and} \quad U \cap V = \emptyset.$$

Let $c \in U$ and $d \in V$. Without loss of generality, assume that $c < d$. Then, define

$$e = \sup \{x \in U : x < d\}.$$

If $e \in U$, then $e < d$, so there exists $x \in U$ such that $e < x < d$. However, this contradicts the definition of e.

On the other hand, if $e \notin U$, then $e \in V$, i.e. there exists $\varepsilon > 0$ such that $(e-\varepsilon, e+\varepsilon) \subseteq V$. Hence, if

$x \in U$ and $x < d$, then $x < e - \varepsilon$ and so,

$$e = \sup\{x \in U : x > d\} < e - \varepsilon$$

which yields a contradiction again.

Example 3.5. If $U,V \subseteq X$ is a separation of X and $Y \subseteq X$ is a connected subspace, prove that

$$Y \subseteq U \quad \text{or} \quad Y \subseteq V.$$

Solution. Since U,V is a separation of X, then we must have

$$X = U \cup V \quad \text{and} \quad U \cap V = \emptyset.$$

Here, U,V are non-empty. Since $Y \subseteq X$ is a connected subspace, then we must have $Y \subseteq U \cup V$. If $Y \subseteq U$, then we must have $Y \subsetneq V$, otherwise it would contradict that $U \cap V = \emptyset$; similarly, if $Y \subseteq V$, then we must have $Y \subsetneq U$, otherwise it would contradict that $U \cap V = \emptyset$ too. □

Proposition 3.2. Let X be a topological space. Then, the following hold:

(i) If $\{A_\alpha\}_{\alpha\in\Lambda}$ is a collection of connected subsets of X such that

$$\bigcap_{\alpha\in\Lambda} A_\alpha \neq \emptyset \quad \text{then} \quad \bigcup_{\alpha\in\Lambda} A_\alpha \subseteq X \text{ is connected}$$

(ii) If $A \subseteq X$ is connected and $A \subseteq B \subseteq \overline{A}$, then B is connected

(iii) If $f : X \to Y$ is continuous and $A \subseteq X$ is connected, then

$$f(A) \subseteq Y \quad \text{is connected.}$$

(iv) If

$$X,Y \text{ are connected} \quad \text{then} \quad X \times Y \text{ is connected.}$$

Definition 3.2 (path and path-connectedness)**.** Given x,y in a topological space X, a path from x to y is

$$\text{a continuous map } f : [a,b] \to X \quad \text{such that} \quad f(a) = x \text{ and } f(b) = y.$$

X is path-connected if for all $x,y \in X$, there exists a path from x to y.

Proposition 3.3. If X is path-connected, then X is connected.

Proof. We shall prove the contrapositive statement instead. Suppose X is not connected. Then, there exists a separation of X, i.e. for any $U,V \subseteq X$, we have

$$U \cup V = X \quad \text{and} \quad U \cap V = \emptyset.$$

Here, U, V are assumed to be non-empty. Let $x \in U \subseteq X$ and $y \in V \subseteq X$ and suppose on the contrary that there exists a path f from x to y, i.e. $f(0) = x$ and $f(1) = y$.

Since f is continuous and $[0,1] \subseteq \mathbb{R}$ is connected, then the image $f([0,1]) \subseteq X$ must also be connected by **(iii)** of Proposition 3.2. So, $f([0,1])$ lies entirely in U or V. However, $U \cap V = \emptyset$, contradicting that $f(0) \in U$ and $f(1) \in V$. As such, no such path f exists. So, f is not path-connected. □

We will see in Example 3.6 (by considering the Topologist's sine curve) that the converse of Proposition 3.3 is not true in general.

Example 3.6 (Topologist's sine curve)**.** Let S be defined as

$$S = \left\{ (x,y) \in \mathbb{R}^2 : y = \sin\left(\frac{2\pi}{x}\right) : 0 < x \leq 1 \right\}.$$

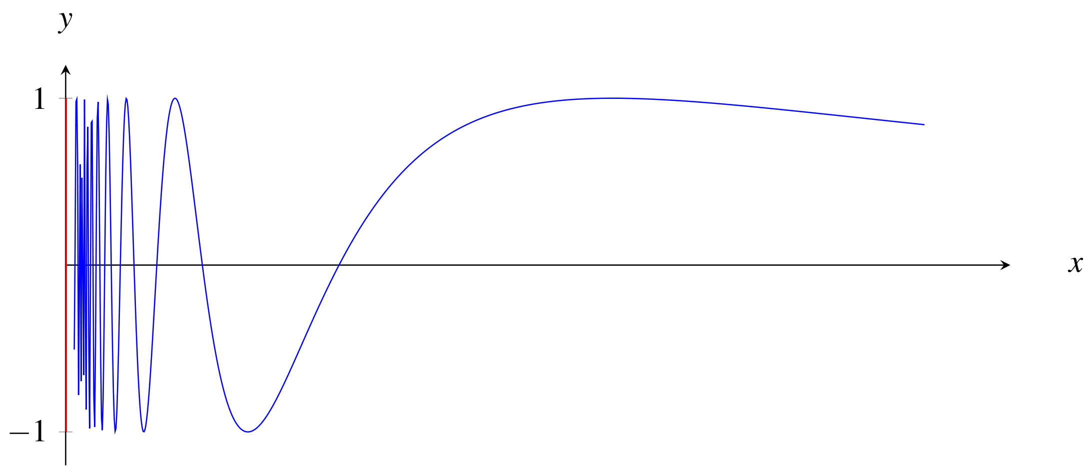


Figure 1: Topologist's sine curve

Let $f : (0,1] \to S$ be defined as

$$t \mapsto \left(t, \sin\left(\frac{2\pi}{t}\right)\right).$$

This gives that $S = f((0,1])$ is path-connected, and thus connected. In particular,

$$\overline{S} = S \cup (\{0\} \times [-1,1]) \quad \text{is connected.}$$

However, $\overline{S}$ is not path-connected! To see why, suppose on the contrary that there exists a path $p : [0,1] \to \overline{S}$ such that

$$p(0) = (0,0) \quad \text{and} \quad p(1) = (1,0).$$

Since $\{0\} \times [-1,1] \subseteq \mathbb{R}^2$ is closed, it shows that $\{0\} \times [-1,1] \subseteq \overline{S}$ is closed. As such,

$$p^{-1}(\{0\} \times [-1,1]) \subseteq [0,1] \quad \text{is closed.}$$

In particular, this shows that $p^{-1}(\{0\}\times[-1,1])\subseteq[0,1]$ has a maximum, which is denoted by b. Consider the restriction of p, i.e.

$$p|_{[b,1]}\to\overline{S}\quad\text{where}\quad t\mapsto(p_1(t),p_2(t)).$$

Then, $p(b)\in\{0\}\times[-1,1]$ but $p((b,1])\subseteq S$. Observe that

$$\lim_{t\to b}p_1(t)=p_1(b)=0\quad\text{and}\quad\text{for all } t>b \text{ we have } p_2(t)=\sin\left(\frac{2\pi}{t}\right)=\sin\left(\frac{2\pi}{p_1(t)}\right).$$

The first statement implies there exists a decreasing sequence $t_i\to b$ such that

$$p_1(t_i)=\frac{1}{i+(-1)^i/4},$$

whereas the second statement implies

$$p_2(t_i)=\sin\left(2\pi i+(-1)^i\cdot\frac{\pi}{2}\right)=(-1)^i.$$

As such,

$$p(t_i)=\left(\frac{1}{i+(-1)^i/4},(-1)^i\right)\quad\text{which does not converge.}$$

3.2. *Connected Components*

Definition 3.3 (connected component). Let X be a topological space. For any $x,y\in X$, define an equivalence relation $\sim$ if there exists a connected subset $C\subseteq X$ such that $x,y\in C$. The equivalence classes of $\sim$ are the connected components of X.

Proposition 3.4. Every connected component of X is connected.

Definition 3.4 (path component). Let X be a topological space. For any $x,y\in X$, define

$$x\overset{p}{\sim}y\quad\text{if there exists a path in } X \quad\text{from } x \text{ to } y.$$

The equivalence classes of $\overset{p}{\sim}$ are called path components.

Definition 3.5 (locally connected). Let X be a topological space and $x\in X$. Then, X is locally connected at x if for all open sets $U\subseteq X$ containing x, there exists a connected open set $V\subseteq X$ such that $x\in V\subseteq U$.

X is locally connected if it is locally connected at every $x\in X$.

Definition 3.5 also holds if we change 'locally connected' to 'locally path-connected'.

Example 3.7. $(0,1)\subseteq\mathbb{R}$ is connected and locally connected.

Example 3.8. $(0,1)\cup(1,2)$ is not connected but locally connected.

Example 3.9. The Topologist's sine curve is connected but not locally connected.

Example 3.10. $\mathbb{Q}\subseteq\mathbb{R}$ is neither connected nor locally connected.

Proposition 3.5. A topological space X is locally connected if and only if for all open $U\subseteq X$, each connected component of U is open in X.

Proposition 3.6. Let X be a topological space. If X is locally path-connected, then the connected components and path components are the same.

3.3. *Countability Axioms*

Definition 3.6 (second countable space)**.** A topological space X is second countable if it has a countable basis.

To put it more precisely, Definition 3.6 means that a topological space X is second countable if there exists some countable collection $\mathcal{U}=\{U_i\}_{i=1}^{\infty}$ of open sets in X such that every open subset of X can be written as a union of elements in $\mathcal{U}$.

Proposition 3.7. X is second countable implies X is first countable.

Example 3.11. $\mathbb{R}^n$ is second countable since

$$\{B_r(\mathbf{x}) : r\in\mathbb{Q},\mathbf{x}\in\mathbb{Q}^n\}\quad \text{is a countable basis for } \mathbb{R}^n.$$

Example 3.12. $\mathbb{R}^\omega$ equipped with the product topology is second countable as

$$\left\{\prod_{n\in\Lambda}(a_n,b_n)\times\prod_{n\in\mathbb{Z}\setminus\Lambda}\mathbb{R} : \Lambda \text{ is finite} : a_n<b_n, a_n,b_n\in\mathbb{Q}\quad \text{for all } n\in\Lambda\right\}$$

is a countable basis.

Definition 3.7 (Lindelöf space)**.** X is a Lindelöf space if every open cover has a countable subcover.

Proposition 3.8. Suppose X is second countable. Then, the following hold:

(i) X is Lindelöf

(ii) There exists a countable subset $A\subseteq X$ that is dense, i.e. $\overline{A}=X$

3.4. *Separation Axioms*

Definition 3.8 (T_3 space). A T_1 topological space X is regular or T_3 if for every $x \in X$ and every closed $B \subseteq X$ such that $x \notin B$, there exist disjoint open sets $U, V \subseteq X$ such that

$$x \in U \quad \text{and} \quad B \subseteq V.$$

Definition 3.9 (T_4 space). A T_1 topological space X is normal or T_4 if for every closed and disjoint $A, B \subseteq X$, there exist disjoint open sets $U, V \subseteq X$ such that

$$A \subseteq U \quad \text{and} \quad B \subseteq V.$$

Remark 3.1. $T_4 \implies T_3 \implies T_2 \implies T_1$

Proposition 3.9. Suppose X is a topological space. X is T_3 if and only if for all $x \in X$, for all $U \subseteq X$ containing x, there exists an open set $V \subseteq X$ containing x such that $\overline{V} \subseteq U$.

Proposition 3.10. Every metrizable space is normal.

Proposition 3.11. If X is a T_3 space with a countable basis, then X is T_4.

Proof. Let $\mathcal{B} \subseteq X$ be a countable basis for X and

$$A \text{ and } B \quad \text{are disjoint closed sets.}$$

We wish to show that A and B have disjoint open neighbourhoods. By the definition of a T_3 space, for every $x \in A$, there exists an open set $U_x \subseteq X$ such that $x \in U$ and $U_x \cap B = \emptyset$.

Since X is T_3, then there exists $V_x \subseteq U_x$ such that V_x is open and $x \in V_x \subseteq \overline{V_x} \subseteq U_x$. Let $W_x \in \mathcal{B}$ such that $x \in W_x \subseteq V_x$ and define the basis $\mathcal{B}'$ to be as follows:

$$\mathcal{B}' = \{W_x \in \mathcal{B} : x \in A\} \quad \text{so} \quad \mathcal{B}' \text{ is a countable subcover.}$$

We have

$$\mathcal{B} \cap \overline{W} = \emptyset \quad \text{for all } W \in \mathcal{B}'.$$

Using a similar method, there exists a countable subset

$$\mathcal{B}'' \subseteq \mathcal{B} \quad \text{such that} \quad A \cap \overline{W} = \emptyset \text{ for all } W \in \mathcal{B}''.$$

Since $\mathcal{B}'$ and $\mathcal{B}''$ are countable, then we can enumerate them as follows:

$$\mathcal{B}' = \{A_1, A_2, \ldots\} \quad \text{and} \quad \mathcal{B}'' = \{B_1, B_2, \ldots\}$$

Let

$$A_n' = A_n \setminus \bigcup_{i=1}^{n} \overline{B_i} \quad \text{and} \quad B_n' = B_n \setminus \bigcup_{i=1}^{n} \overline{A_i}.$$

Then, define

$$U_A = \bigcup_{i=1}^{\infty} A_n' \quad \text{and} \quad U_B = \bigcup_{i=1}^{\infty} B_n'.$$

Recall that $A \cap \overline{B_i} = \emptyset$ for all i. Hence, $A \cap A_i = A \cap A_i'$, which implies

$$A = \bigcup_{i=1}^{\infty} (A \cap A_i) = \bigcup_{i=1}^{\infty} \left(A \cap A_i'\right) \subseteq \bigcup_{i=1}^{\infty} A_i' = U_A.$$

Suppose there exists $x \in U_A \cap U_B$. Then, there exists $n, m \in \mathbb{N}$ such that $x \in A_n' \cap B_m'$. Hence,

$$x \in A_n \quad \text{and} \quad x \notin \bigcup_{i=1}^{n} \overline{B_i}.$$

Concurrently, we also have

$$x \in B_n \quad \text{and} \quad x \notin \bigcup_{i=1}^{m} \overline{A_i}.$$

However, since $x \in A_n$ and $x \in B_m$, it implies that $x \in A \cap B$, contradicting the fact that $A \cap B = \emptyset$. We conclude that $U_A \cap U_B = \emptyset$. □

4. Urysohn's Metrization Theorem and Tychonoff's Theorem

4.1. *Urysohn's Metrization Theorem*

Theorem 4.1 (Urysohn metrization theorem)**.** If X is a regular topological space with a countable basis, then it is metrizable.

Definition 4.1. Let $A, B \subseteq X$. We say that A and B are separated by a continuous function if

$$\text{there exists a continuous } f : X \to [0,1] \quad \text{such that} \quad f(A) = 0 \text{ and } f(B) = 1.$$

Example 4.1. An example of two sets A and B separated by a continuous function can be constructed in $\mathbb{R}$. Consider the sets $A = [0,1]$ and $B = [2,3]$.

Define the continuous function $f : \mathbb{R} \to [0,1]$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1]; \\ 1 & \text{if } x \in [2,3]; \\ x-1 & \text{if } x \in (1,2) \end{cases} \quad \text{which is continuous on } \mathbb{R}.$$

Moreover, $f(A) = 0$ and $f(B) = 1$, and thus separates the sets A and B.

Definition 4.2 ($T_{3\frac{1}{2}}$ space)**.** X is completely regular or $T_{3\frac{1}{2}}$ if it is T_1 and for every $x \in X$ and closed $A \subseteq X$ such that $x \notin A$, we have

$$\{x\} \text{ and } A \quad \text{are separated by a continuous function.}$$

Definition 4.3 (T_5 space)**.** X is completely normal or T_5 if it is T_1 and for every disjoint closed sets A and B, we have

$$A \text{ and } B \quad \text{are separated by a continuous function.}$$

Example 4.2. X is $T_{3\frac{1}{2}}$ implies X is T_3.

Example 4.3. X is T_5 implies X is T_4.

Lemma 4.1 (Urysohn's lemma)**.** Let X be a T_4 space and A, B be closed disjoint subsets of X. Let $[a,b]$ be a closed interval of $\mathbb{R}$. Then, there exists a continuous map $f : X \to [a,b]$ such that

$$f(x) = \begin{cases} a & \text{for every } x \in A; \\ b & \text{for every } x \in B. \end{cases}$$

Definition 4.4 (topological embedding)**.** Let X, Y be two topological spaces and $f : X \to Y$ be an injective continuous map. We say that

$$f \text{ is a topological embedding} \quad \text{if} \quad f \text{ is a homeomorphism between } X \text{ and } f(X).$$

Lemma 4.2. Let X be a topological space. Suppose $\{f_\alpha\}_{\alpha\in\Lambda}$ is a family of continuous functions from X to $\mathbb{R}$ satisfying the following property: for all $x \in X$ and open $U \subseteq X$ such that $x \in U$, there exists $\alpha \in \Lambda$ such that

$$f_\alpha(x) > 0 \quad \text{and} \quad f_\alpha(X \backslash U) = \{0\}.$$

Then, the map

$$F : X \to \mathbb{R}^\Lambda \text{ where } x \mapsto (f_\alpha(x))_{\alpha\in\Lambda} \quad \text{is an embedding of } X \text{ into } \mathbb{R}^\Lambda.$$

4.2. *Tychonoff's Theorem*

Theorem 4.2 (Tychonoff's theorem)**.** The product of compact spaces is compact.

Note that Tychonoff's theorem (Theorem 4.2) is equivalent to saying the following: if $\{X_\alpha\}_{\alpha\in\Lambda}$ is a family of compact spaces, then

$$X = \prod_{\alpha\in\Lambda} X_\alpha \text{ is compact} \quad \text{with respect to the product topology.}$$

5. The Arzela-Ascoli Theorem

5.1. *The Compact-Open Topology*

5.2. *Equicontinuity*